Lecture 3
Dynamic Equilibrium Models III: Infinite Periods

1. Introduction

In this lecture, we extend our analysis to infinite periods. The method of dynamic programming can be easily applied to solve infinite horizon optimization problems. In fact, in certain cases solving infinite horizon DGE models is easier than solving finite horizon DGE models. For expository simplicity, we will focus on stationary problems. The assumption of stationarity implies that the period utility function and the laws of motion are time-invariant functions i.e.

$$F_t(X_t, U_t) = F(X_t, U_t) \& x_{t+1} = m_x(X_t, U_t) \forall t.$$  \hspace{1cm} (1.1)

Time-invariant period utility function and laws of motion generate time-invariant (or stationary) policy and value functions. Thus,

$$U_t = G(X_t) \& W_t(X_t) = W(X_t) \forall t.$$  \hspace{1cm} (1.2)

We will also assume that period return function is bounded 0 ≤ F(X_t, U_t) < ∞ and \lim_{t \to \infty} \beta^t F(X_t, U_t) = 0. These assumptions ensure that we get interior solutions. All the examples considered in this course will satisfy these assumptions.

Suppose that the optimization problem is as follows:

$$W = \max_{U_t} \sum_{t=0}^{\infty} \beta^t F(X_t, U_t)$$  \hspace{1cm} (1.3)

subject to the laws of motion

$$x_{t+1} = m_x(X_t, U_t) \forall x_t \in X_t \& \forall t$$  \hspace{1cm} (1.4)

given X_0. We can recast this optimization problem as

$$W(X_t) = \max_{U_t} F(X_t, U_t) + \beta W(X_{t+1})$$  \hspace{1cm} (1.5)

subject to the laws of motion

$$x_{t+1} = m_x(X_t, U_t) \forall x_t \in X_t$$  \hspace{1cm} (1.6)

With stationary problems, it is customary to drop the time subscript. The optimization problem is normally written as

$$W(X) = \max_{U} F(X, U) + \beta W(X')$$  \hspace{1cm} (1.5a)

subject to the laws of motion
\[ x' = m_x(X, U) \forall x \in X \quad (1.6a) \]

where terms like \( X' \) denote next period values of state variables. Potentially we can solve such Bellman equations in four ways:

1. guess and verify the policy function;
2. guess and verify the value function;
3. repeated substitution; and
4. through iteration of value function.

Guess and verify methods are applicable to very limited type of cases. Essentially, these methods work for only two classes of specifications of preferences and constraints, namely, variants of specification with linear constraints and quadratic preferences or Cobb-Douglas constraints and logarithmic preferences. More often than not one has to use either method 3 or 4. Also only in the limited cases, dynamic programming problems can be solved analytically. Generally, one uses approximation and/or numerical methods to solve dynamic programming problems.

Example 1

Let us illustrate the four methods of solution using a concrete example. Consider infinite horizon version of optimal consumption problem with production. The optimization problem is

\[ \max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln c_t \quad (1.7) \]

subject to

\[ k_{t+1} = k_t^\alpha - c_t \quad (1.8) \]

We can recast the above optimization problem as

\[ W(k_t) = \max_{c_t, k_{t+1}} \ln c_t + \beta W(k_{t+1}) \quad (1.9) \]

subject to (1.8).

Method 1: Guess and Verify Policy Function

Let \( \lambda_t \) be the Langrangian multiplier associated with (1.8), then first order conditions are

\[ c_t : \quad \frac{1}{c_t} = \lambda_t \quad (1.10) \]
\[ k_{t+1} : \beta \frac{dW(k_{t+1})}{dk_{t+1}} = \lambda_t \] (1.11)

From envelope condition we have

\[ \frac{dW(k_t)}{dk_t} = \frac{\alpha k_t^{\alpha-1}}{c_t} \] (1.12)

Combining (1.10-1.12) we have

\[ \frac{1}{c_t} = \frac{\alpha \beta k_t^{\alpha-1}}{c_{t+1}}. \] (1.13)

(1.13) is the Euler equation linking consumptions in adjacent periods. In order to find the path of optimal consumption, we need to solve (1.13). We will use guess and verify method to solve this equation. Suppose that the policy function has the following form:

\[ c_t = \mu k_t^\alpha \] (1.14)

Basically we are assuming that each period the decision maker consumes a constant fraction of income. The trick is to find an expression for \( \mu \) which satisfies (1.13). Putting (1.14) in (1.13), we get

\[ \frac{1}{\mu k_t^\alpha} = \frac{\alpha \beta k_t^{\alpha-1}}{\mu k_{t+1}^{\alpha}}. \] (1.15)

From (1.15) we have

\[ \frac{1}{k_t^\alpha} = \frac{\alpha \beta}{k_{t+1}}. \] (1.16)

This implies

\[ k_{t+1} = \alpha \beta k_t^\alpha \] (1.17)

Putting (1.14) and (1.17) in the budget constraint we have

\[ k_t^\alpha = \mu k_t^\alpha + \alpha \beta k_t^\alpha \] (1.18)

(1.18) implies that \( \mu = 1 - \alpha \beta \). The solution for optimal consumption is then

\[ c_t = (1 - \alpha \beta)k_t^\alpha \] (1.19)

**Steps**

(1) Set up the Bellman equation;
(2) Derive first order conditions and the Euler equations;
(3) Guess the functional form of the policy functions;
(4) Putting the guessed policy function in the Euler equation solve for the coefficients.

**Method 2. Guess and Verify Value Function**

Let us suppose that the value function has the following form:

\[ W(k_t) = E + F \ln k_t \]  

(1.20)

We basically want to derive expressions for \( E \) and \( F \). The guess depends on the form of utility function and the number of state variables. With this guess the optimization problem is

\[ W(k_t) = \max_{c_t, k_{t+1}} \ln c_t + \beta E + \beta F \ln k_{t+1} + \lambda_t [k_t^\alpha - c_t - k_{t+1}] \]  

(1.21)

The first order conditions are:

\[ c_t : \quad \frac{1}{c_t} = \lambda_t \]  

(1.22)

\[ k_{t+1} : \quad \frac{\beta F}{k_{t+1}} = \lambda_t. \]  

(1.23)

(1.22), (1.23), and the budget constraint imply

\[ \frac{\beta F}{k_t^\alpha - c_t} = \frac{1}{c_t}. \]  

(1.24)

This implies

\[ c_t = \frac{1}{1 + \beta F k_t^\alpha}. \]  

(1.25)

Also

\[ k_{t+1} = \frac{\beta F}{1 + \beta F k_t^\alpha}. \]  

(1.26)

Putting (1.25) and (1.26) in (1.21) we get

\[ W(k_t) = \ln \left( \frac{1}{1 + \beta F k_t^\alpha} \right) + \beta E + \beta F \ln \left( \frac{\beta F}{1 + \beta F k_t^\alpha} \right). \]  

(1.27)

(1.27) implies that

\[ W(k_t) = \text{some constant terms} + \alpha (1 + \beta F) \ln k_t. \]  

(1.28)

Comparing (1.20) and (1.28) we have

\[ E = \text{some constant terms} \]  

(1.29)

and
\[ F = \frac{\alpha}{1 - \alpha \beta} \]  

(1.30)

(1.25) and (1.30) imply that  

\[ c_t = (1 - \alpha \beta) k_t^\alpha \]  

(1.31)

which is identical to (1.19).

**Steps**

1. Guess the form of the value function;
2. Set up Bellman equation;
3. Derive first order conditions and solve for the policy functions;
4. Put the derived policy functions in the value function;
5. Compare the new value function with the guessed one and solve for the coefficients.

**Method 3. Use of Envelope Condition and Repeated Substitution**

We go back to Euler equation (1.13)

\[ \frac{1}{c_t} = \frac{\alpha \beta k_t^{\alpha - 1}}{c_{t+1}}. \]  

(1.13)

Previously we solved this equation by guessing the form of policy function. Now we are going to solve this equation another way. We know that \((1 + r) = \alpha k^{\alpha - 1}\). Using this fact, we can rewrite (1.13) as

\[ c_{t+1} = \beta(1 + r_{t+1})c_t. \]  

(1.32)

Making use of (1.32) we have

\[ c_{t+j} = \beta^j \prod_{i=1}^j (1 + r_{t+i}) c_t. \]  

(1.33)

The budget constraint is

\[ k_t^\alpha = c_t + k_{t+1} \]  

(1.34)

which can be rewritten as

\[ \frac{\alpha k_t^{\alpha - 1}}{\alpha} k_t = c_t + k_{t+1}. \]  

(1.35)

From (1.35) we have another expression for \(k_t\)

\[ k_t = \frac{\alpha}{1 + r_t} [c_t + k_{t+1}]. \]  

(1.36)
Now we solve forward (1.36).

\[ k_t = \frac{\alpha}{1 + r_t} c_t + \frac{\alpha^2}{(1 + r_t)(1 + r_{t+1})} c_{t+1} + \frac{\alpha^3}{(1 + r_t)(1 + r_{t+1})(1 + r_{t+2})} c_{t+2} \]  

(1.37)

Solving forward (1.37) repeatedly and assuming that \( \lim_{T \to \infty} \frac{\alpha T+1}{\prod_{j=0}^{T-1} (1 + r_{t+j})} k_T = 0 \), we have

\[ k_t = \frac{\alpha}{1 + r_t} c_t + \frac{\alpha^2 \beta}{1 + r_t} c_t + \frac{\alpha^3 \beta^2}{1 + r_t} c_t + \ldots \]  

(1.38)

Combining (1.33) with (1.38) we have

\[ k_t = \frac{\alpha}{1 + r_t} c_t + \frac{\alpha^2 \beta}{1 + r_t} c_t + \frac{\alpha^3 \beta^2}{1 + r_t} c_t + \ldots \]  

(1.39)

(1.39) can be rewritten as

\[ \frac{1 + r_t}{\alpha} k_t = [1 + \alpha \beta + \alpha^2 \beta^2 + \ldots] c_t = \frac{1}{1 - \alpha \beta} c_t. \]  

(1.40)

From (1.40) we again have

\[ c_t = (1 - \alpha \beta) k_t^\alpha. \]  

(1.41)

Steps

(1) Steps 1-3 as in the method 1;
(2) Using the Euler equations solve for the future values of choice variables as functions of the current values of choice variables;
(3) Solve forward the laws of motion;
(4) Combine expressions found in steps 3 and 4.

Method 4. Value Function Iteration

The optimization problem is

\[ W(k_t) = \max_{c_t, k_{t+1}} \ln c_t + \beta W(k_{t+1}) \]  

subject to

\[ k_{t+1} = k_t^\alpha - c_t. \]  

(1.43)

Start at some time \( T \) and assume that \( k_{T+1} \) & \( W_{T+1}(k_{T+1}) = 0 \). Actually you can assume any finite value for \( k_{T+1} \) & \( W_{T+1}(k_{T+1}) \). Then at time \( T \) the optimization problem reduces to
\[ W_T(k_T) = \max_{c_T} \ln c_T + \lambda_T[k_T^\alpha - c_T]. \] (1.44)

The solution is
\[ c_T = k_T^\alpha. \] (1.45)

The value function
\[ W_T(k_T) = \alpha \ln k_T. \] (1.46)

Now go back to period \( T - 1 \). The optimization problem is
\[ W_{T-1}(k_{T-1}) = \max_{c_{T-1}, k_T} \ln c_{T-1} + \alpha \beta \ln k_T + \lambda_{T-1} \left[k_{T-1}^\alpha - c_{T-1} - k_T\right] \] (1.47)

The first order conditions are
\[ c_{T-1} : \frac{1}{c_{T-1}} = \lambda_{T-1} \] (1.48)
\[ k_T : \frac{\alpha \beta}{k_T} = \lambda_{T-1}. \] (1.49)

From (1.48), (1.49), and the period budget constraint we have
\[ \frac{\alpha \beta}{k_T} = \frac{1}{k_{T-1}^\alpha - k_T}. \] (1.50)

From (1.50) we have
\[ k_T = \frac{\alpha \beta}{1 + \alpha \beta k_{T-1}^\alpha}. \] (1.51)

From the budget constraint we get
\[ c_{T-1} = \frac{1}{1 + \alpha \beta k_{T-1}^\alpha}. \] (1.52)

Using (1.51) and (1.52) the value function for the period \( T - 1 \) can be written as
\[ W_{T-1}(k_{T-1}) = \text{constant terms} + \alpha(1 + \alpha \beta) \ln k_{T-1}. \] (1.53)

Now we go back to period \( T - 2 \). The optimization problem now is
\[ W_{T-2}(k_{T-2}) = \max_{c_{T-2}, k_{T-1}} \ln c_{T-2} + \beta \text{ cons.} + \alpha \beta(1 + \alpha \beta) \ln k_{T-1} + \lambda_{T-2} \left[k_{T-2}^\alpha - c_{T-2} - k_{T-1}\right] \] (1.54)

You can show that the solutions are
\[ k_{T-1} = \frac{\alpha \beta + \alpha^2 \beta^2}{1 + \alpha \beta + \alpha^2 \beta^2} k_{T-2} \]  \hspace{1cm} (1.55)

\[ c_{T-2} = \frac{1}{1 + \alpha \beta + \alpha^2 \beta^2} k_{T-2}^\alpha. \]  \hspace{1cm} (1.56)

Using (1.55) and (1.56) we can get expression for \( W_{T-2}(k_{T-2}) \). Then we step back one more period and solve for \( W_{T-3}(k_{T-3}) \). We keep on repeating these steps for a large number of periods. One can show that

\[ c_{T-j} = \frac{1}{1 + \alpha \beta + \alpha^2 \beta^2 + \ldots + \alpha^j \beta^j} k_{T-j}^\alpha. \]  \hspace{1cm} (1.57)

Then

\[ \lim_{j \to \infty} c_{T-j} \equiv c_t = (1 - \alpha \beta) k_t^\alpha. \]  \hspace{1cm} (1.58)

The value function converges too.

**Steps**

1. Start at some period \( T \) and assume that \( k_{T+1} \& W_{T+1}(k_{T+1}) = 0 \);
2. Set up the Bellman equation for period \( T \);
3. Derive the policy functions;
4. Using the policy functions, derive the value function for \( T \);
5. Step 1 period back and repeat steps 2-4;
6. Keep on repeating step 4 till convergence is achieved in policy and value functions.

**2. Uncertainty**

The method of dynamic programming is readily applicable to infinite horizon optimization problems with uncertainty. Potentially one can use all the four methods discussed above to solve stochastic dynamic problems. We begin with a concrete example.

**Example 2**

*Real Business Cycle (RBC) Model*

We are going to analyze business cycle using the RBC model. This model has been very influential in studying business cycles. More generally, it has changed the methodology of macroeconomics and led to emergence of Dynamic Stochastic General Equilibrium (DSGE) models. Most of the analysis in modern macroeconomics is done using this framework. As discussed earlier, the DSGE models have the following properties: (i) They specify budget constraints for households, technologies for firms, and resource constraints for the overall economy; (ii) They specify household preferences and firm objectives; (iii) They assume
forward-looking behavior for firms and households; (iv) They include the shocks that firms and households face; (v) They are models of the entire economy.

Before developing the RBC model, we summarize the most important business cycle facts. Firstly, empirical evidence suggests that consumption, output, investment, and employment are highly pro-cyclical. Real wage is acyclical or mildly pro-cyclical. Secondly, the effect of temporary shocks on output is highly persistent. The effect of temporary shocks last up to eight quarters. Thirdly, the response of output to a shock is hump-shaped. The effect of shock on output peaks in fourth or fifth quarter. Fourthly, consumption is less volatile relative to output, while investment is highly volatile relative to output. Any good model of business cycle should be able to to match and explain these facts.

Broadly, there are two approaches to study business cycles: (i) RBC models and (ii) Keynesian Models. There are two key differences between these two types of approaches. RBC models assume Walrasian markets and they attribute business cycles primarily to technology (productivity/real/supply side) shocks. Keynesian models on the other hand assume imperfect markets with nominal rigidities and they attribute business cycles primarily to aggregate demand or nominal shocks. We studied Keynesian models in ECON 501. In this lecture, we will develop RBC model and evaluate its performance in explaining business cycle regularities.

The RBC model is a stochastic version of the optimal consumption problem analyzed above. In the basic RBC model, it is assumed that there is uncertainty with regard to technology or production function. This model brings out the effects of technology shock on consumption, output, employment etc.

Consider a stochastic version of the optimal consumption problem analyzed above. Suppose that the production function is given by \( A_t k_t^\alpha \) where \( A_t = \exp^{\epsilon_t} \). \( \epsilon_t \) is an independently and identically distributed (i.i.d) random variable with mean 0 and variance \( \sigma^2 \). The random variable \( A_t \) is called technology or productivity shock. It is assumed that the technology shock is realized at the beginning of period \( t \) before consumption and investment decisions are made. Let \( E_t \) denote the expectation operator conditional on time \( t \) information set.

The optimization problem is

\[
\max_{c_t, k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t
\]

subject to

\[
k_{t+1} = A_t k_t^\alpha - c_t.
\]

This optimization problem is known as the real business cycle model, which studies the effects of technology shocks on investment, consumption, income etc. We can recast the above optimization problem as

\[
W(k_t) = \max_{c_t, k_{t+1}} \ln c_t + \beta E_t W(k_{t+1}) + \lambda_t [A_t k_t^\alpha - c_t - k_{t+1}].
\]

The first order conditions are
\[ c_t : \frac{1}{c_t} = \lambda_t \] (2.4)

\[ k_{t+1} : \beta E_t \frac{dW(k_{t+1})}{dk_{t+1}} = \lambda_t. \] (2.5)

From envelope condition we have

\[ dW(k_t) \frac{dk_t}{dt} = \frac{\alpha A_t k_t^{\alpha-1}}{c_t}. \] (2.6)

Combining (2.4)-(2.6) we get

\[ \frac{1}{c_t} = \alpha \beta E_t \frac{A_{t+1} k_{t+1}^{\alpha-1}}{c_{t+1}}. \] (2.7)

which is again the Euler equation. In order to solve (2.7), we will guess the policy function (you can use other methods as well). Let

\[ c_t = \mu A_t k_t^\alpha. \] (2.8)

Putting (2.8) in (2.7) we have

\[ \frac{1}{A_t k_t^\alpha} = \alpha \beta E_t \frac{1}{k_{t+1}}. \] (2.9)

Combining (2.9) with the budget constraint we have

\[ \frac{1}{A_t k_t^\alpha} = \frac{\alpha \beta}{A_t k_t^\alpha - c_t}. \] (2.10)

From (2.10) we get

\[ c_t = (1 - \alpha \beta) A_t k_t^\alpha. \] (2.11)

From the budget constraint we get

\[ k_{t+1} = \alpha \beta A_t k_t^\alpha. \] (2.12)

Taking log of (2.12) we get fundamental stochastic difference equation which tells us how capital stock evolves over time.

\[ \ln k_{t+1} = \ln \alpha \beta + \alpha \ln k_t + \epsilon_t \] (2.13)

Using (2.13) we can trace out how capital accumulation evolves over time in response to a single shock, \( \epsilon_t \) (impulse response function). One can also derive moments of the process of capital accumulation. Solving (2.13) backwards we have

\[ \ln k_{t+1} = \ln \alpha \beta + \alpha [\ln \alpha \beta + \alpha k_{t-1} + \epsilon_{t-1}] + \epsilon_t. \] (2.14)
If we keep on repeating this process, we will get

$$\ln k_{t+1} = [1 + \alpha + \alpha^2 + ...] \ln \alpha \beta + [\epsilon_t + \alpha \epsilon_{t-1} + .......]$$  \hfill (2.15)

Then

$$E(\ln k_{t+1}) = \frac{\ln \alpha \beta}{1 - \alpha}$$ \hfill (2.16)

$$V(\ln k_{t+1}) = \frac{\sigma^2}{1 - \alpha^2}.$$ \hfill (2.17)

Similarly we can derive moments of other variables like consumption, income etc., covariances, and the associated impulse response functions.

**Exercise:** Guess the form of the value function as $W(k_t, A_t) = E + F \ln k_t + G \ln A_t$ and solve the above example.

Denote output by $y_t$. Then, we have

$$\ln y_t = \ln A_t + \alpha \ln k_t.$$  \hfill (2.18)

By combining (2.13) and (2.18) we have,

$$\ln k_t = \ln \alpha \beta + \ln y_{t-1}.$$ \hfill (2.19)

Then (2.18) and (2.19) imply that log of output follows a first order autoregressive process:

$$\ln y_{t+1} = \alpha \ln \alpha \beta + \alpha \ln y_t + \xi_{t+1}.$$ \hfill (2.20)

Let us now consider implications of technology shock on consumption, investment, and output. Using (2.20) one can derive the implications of technology shock on output. For this one needs to assume some value of $\alpha$. $\alpha$ is estimated to be 0.33.

Now consider the effect of a one time positive technology shock. Let $y_0$ be the output at time $t - 1$. Suppose that at time $T$ technology shock is realized and let $\xi_T = 1$. Suppose that technology shock is entirely temporary, i.e. $\xi_t = 0$, $\forall t > T$. What would be the response of output. In period $T$ log of output will be one unit higher than log of $y_0$. In $T + 1$ it will be higher by $1/3$ compared to log of $y_0$. In $T + 2$ it will be higher by $1/9$ and so on.

There are two problems with this kind of response. First, response of output is not hump shaped as in the data. Output rises in period $T$ and then falls linearly. Secondly, the effect of technology shock is not persistent. Its effect dies down quite quickly. There are other problems as well. In the model, consumption and investment are a constant proportion of output. Thus, both are as volatile as output. However, in the real data consumption is much less volatile than output and investment is much more volatile. This analysis suggests that this version of RBC model does not do a good job in accounting for business cycle facts.
Now let us modify the environment as follows. Suppose now that technology shock follows an autoregressive process:

\[ \xi_{t+1} = \rho \xi_t + u_{t+1} \]  

where \( u_t \) is i.i.d. with mean zero and variance \( \sigma^2 \). Then (2.20) can be written as

\[ \ln y_{t+1} = \alpha \ln \beta + \alpha \ln y_t + \rho \xi_t + u_{t+1}. \]  

(2.20) also implies that

\[ \xi_t = \ln y_t - \alpha \ln \beta - \alpha \ln y_{t-1}. \]  

Then (2.23) and (2.24) imply that log of output follows a second order autoregressive process:

\[ \ln y_{t+1} = \text{Constnat} + (\alpha + \rho) \ln y_t - \alpha \rho \ln y_{t-1} + u_{t+1}. \]  

(2.24) shows that the response of output to technology shock depends on two parameters, \( \alpha \) and \( \rho \). If \( \rho \) is high, meaning that technology shock is highly persistent, the model can generate a hump-shaped and persistent response of output to technology shock. However, it does not fix other problems.

The models considered so far do not say anything about employment and real wage. To consider the implications with regard to employment and real wage, let us introduce labor-leisure choice in example 2. Let \( n_t \) be labor supplied at time \( t \). Now the problem is

\[
\max_{c_t,n_t,k_{t+1}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \left\{ \ln c_t + \ln (1 - n_t) \right\} \right]
\]

subject to

\[ A_t k_t^{\alpha} n_t^{1-\alpha} = c_t + k_{t+1}. \]  

The Bellman equation is

\[ W(k_t) = \max_{c_t,n_t,k_{t+1}} \ln c_t + \ln (1 - n_t) + \beta E_t W(k_{t+1}) + \lambda_t \left[ A_t k_t^{\alpha} n_t^{1-\alpha} - c_t - k_{t+1} \right]. \]

The first order conditions for consumption continues to be given by (2.4). The first order conditions for labor supply and capital stock are

\[ n_t : \frac{1}{1 - n_t} = \lambda_t (1 - \alpha) A_t k_t^{\alpha} n_t^{-\alpha}; \]

and

\[ k_{t+1} : \lambda_t = \beta \alpha E_t \lambda_{t+1} A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}. \]
(2.4) and (2.28) imply that
\[ \frac{1}{c_t} = \alpha \beta \frac{E_t A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}}{c_{t+1}}. \] (2.30)
which is the Euler equation linking consumption in adjacent periods. Using guess and verify method, one can show that
\[ c_t = (1 - \alpha \beta) A_t k_t^\alpha n_t^{1-\alpha}; \] (2.31)
\[ k_{t+1} = \alpha \beta A_t k_t^\alpha n_t^{1-\alpha} \] (2.32)
and
\[ n_t = \frac{1 - \alpha}{2 - \alpha (1 + \beta)}. \] (2.33)
Since \( n_t \) is constant, the model implies that technology shock does not affect employment. Also since real wage is equal to the marginal product of labor \( (= (1 - \alpha) y_t / n_t) \), real wage is strongly pro-cyclical. Both these predictions are at odds with empirical evidence. As discussed earlier, employment is highly pro-cyclical and real wage is essentially acyclical.

What we have seen that basic RBC models cannot account for most of the business cycle facts. Researchers have modified basic RBC models in many ways to improve the performance of these models. Some of the most prominent extensions are introduction of government expenditure and fiscal shocks, taste shocks (shocks to utility function), more complicated utility function (habit formation), shocks to inventory etc. Many researchers have taken more fundamental departures abandoning the assumptions of Walrasian or frictionless markets. Now, it is customary to assume friction in either labor market or goods market or both. Frictions in the labor market generate unemployment and have allowed researchers to also analyze unemployment dynamics. Some models also incorporate credit markets, but in very rudimentary forms. These extensions have improved the performance of these models in some dimensions, but not to a great extent. Of course, none of these models were able to forecast the current great recession.
Additional Exercises

(1) Brock-Mirman Optimal Growth Model: Suppose that the representative agent faces the following optimization problem

\[ \max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \ln c_t \]

subject to

\[ k_t^\alpha [\lambda^tn_t]^{1-\alpha} = c_t + k_{t+1} \]

where the labor force \( n_t = 1 \) and \( \lambda > 1 \) is the rate of labor augmenting technical progress. Solve for the optimal path of consumption, income, and real wages.

(2) Endogenous Labor Supply: Suppose now that the representative agent faces the following optimization problem

\[ \max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \ln c_t + \ln(1 - n_t) \]

subject to

\[ y_t = c_t + k_{t+1}. \]

(a.) Let \( y_t = k_t^\alpha n_t^{1-\alpha} \). Solve for the optimal path of consumption, income, employment, and real wages.

(b.) Let \( y_t = k_t^\alpha [\lambda^tn_t]^{1-\alpha} \). Solve for the optimal path of consumption, income, employment, and real wages.

(3) Government Expenditure and the Business Cycle: Suppose that the representative agent faces the following optimization problem

\[ \max_{c_t, k_{t+1}} E_0 \sum_{t=0}^{\infty} \ln c_t + \ln(1 - n_t) \]

subject to

\[ y_t = c_t + k_{t+1} + g_t. \]

Let \( y_t = k_t^\alpha n_t^{1-\alpha} \). Also suppose that the government expenditure \( g_t = \theta y_t \). Assume that the representative agent ignores the effect of his investment decision on the government expenditure (an externality). Solve for the optimal path of consumption, income, employment, and real wages. How do changes in \( \theta \) affect the labor supply?
(4) Let the optimization problem be

\[
\max_{c_t, a_{t+1}} \sum_{t=0}^{\infty} \frac{c_t^{1-\alpha}}{1-\alpha}
\]

subject to

\[(1 + r) a_t = c_t + a_{t+1}.\]

Solve for the optimal path of \(c_t\) and \(a_{t+1}\).

(5) Let us introduce uncertainty in the previous problem. Let the optimization problem be

\[
\max_{c_t, a_{t+1}} E_0 \sum_{t=0}^{\infty} \frac{c_t^{1-\alpha}}{1-\alpha}
\]

subject to

\[a_{t+1} = (1 + r_{t+1})(a_t - c_t)\]

where \(r_t\) is an i.i.d random variable. Solve for the optimal path of \(c_t\) and \(a_{t+1}\).