Lecture 3
The Real Business Cycle Model

1. Introduction

In the previous lecture, we analyzed two-period models. In this lecture, we extend our analysis to infinite periods. In particular, we develop infinitely-lived representative agent models. These models are commonly used to analyze macro issues. We first consider the optimal consumption/savings problem in a model with the perfect foresight (no uncertainty). Then we extend the model to incorporate uncertainty. The introduction of uncertainty will allow us to study business cycles. In the second part, we will develop the Real Business Cycle (RBC) model, which is widely used to study business cycles.

Example 1

Consider infinite horizon version of the optimal consumption problem with production. The main difference from the two-period model we considered in the previous lecture is that now agents are infinitely lived. As before consumers own firms and all markets are competitive.

The optimization problem faced by the representative agent is to

\[
\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln c_t
\]

subject to

\[
k_{t+1} = k_t^\alpha - c_t, \quad \forall t.
\]

We can rewrite the problem as

\[
\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t [\ln c_t + \lambda_t (k_t^\alpha - c_t - k_{t+1})]
\]

where \(\lambda_t\) is the Langrangian multiplier associated with the resource constraint. The first order conditions are

\[
c_t : \frac{1}{c_t} = \lambda_t
\]
and

\[ k_{t+1} : \beta^t \lambda_t = \beta^{t+1} \lambda_{t+1} \alpha k_{t+1}^{\alpha-1}. \]  

(5)

Combining (4) and (5), we have

\[ \frac{1}{c_t} = \frac{\alpha \beta k_{t+1}^{\alpha-1}}{c_{t+1}}. \]  

(6)

(6) is the Euler equation linking consumptions in adjacent periods. In order to find the path of optimal consumption, we need to solve (6).

We will use guess and verify method or the method of undetermined coefficients to solve this equation. As the name suggests, this method makes a guess about the optimal path of consumption, \( c_t \), and then verifies whether the assumed path satisfies (6) or not.

Suppose that the optimal path of consumption has following form:

\[ c_t = \mu k_t^\alpha \]  

(7)

where \( \mu \) is an unknown constant. Basically we are assuming that each period the decision maker consumes a constant fraction of output. The trick is to find an expression for \( \mu \) which satisfies (6). Putting (7) in (6), we get

\[ \frac{1}{\mu k_t^\alpha} = \frac{\alpha \beta k_{t+1}^{\alpha-1}}{\mu k_{t+1}^\alpha}. \]  

(8)

From (8) we have

\[ \frac{1}{k_t^\alpha} = \frac{\alpha \beta}{k_{t+1}^\alpha}. \]  

(9)

This implies

\[ k_{t+1} = \alpha \beta k_t^\alpha. \]  

(10)

Putting (7) and (10) in the budget constraint we have

\[ k_t^\alpha = \mu k_t^\alpha + \alpha \beta k_t^\alpha. \]  

(11)

(11) implies that \( \mu = 1 - \alpha \beta \). The solution for optimal consumption is then

\[ c_t = (1 - \alpha \beta) k_t^\alpha. \]  

(12)
(10) and (12) show that investment and consumption are a constant proportion of output. This result is due to our assumptions of logarithmic utility function, Cobb-Douglas production function, and 100% depreciation rate. Recall that the Solow model assumes that agents in the economy save a constant proportion of the GDP. This example provides a model in which this is true.

To solve this model, we used guess and verify method. However, there are limitations to this method. These methods work for only two classes of specifications of preferences and constraints, namely, variants of specification with linear constraints and quadratic preferences or Cobb-Douglas constraints and logarithmic preferences. Also only in limited cases, the dynamic general equilibrium models can be solved analytically. Generally, one uses approximation and/or numerical methods to solve these models. Learning these methods is beyond the scope of this course. We will only consider examples in which guess and verify method works. The interested readers may go to the website of Dynare.Org (http://www.dynare.org/). The website provides a rich set of programs, software tools, and methods used to solve DGE models.

Exercise: Derive steady state values of consumption, capital stock, and the real interest rate.

Example 2

Next we consider a model in which labor supply is endogenous. Suppose that labor market is competitive. Let \( n_t \) denote the labor supplied by the representative agent at time \( t \). Now the representative agent faces the following optimization problem

\[
\max_{c_t, n_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t [\ln c_t + \ln(1 - n_t)]
\]

subject to

\[
k_t^\alpha n_t^{1-\alpha} = c_t + k_{t+1}, \ \forall t.
\]

Again we can recast the problem as

\[
\max_{c_t, n_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t [\ln c_t + \ln(1 - n_t) + \lambda_t[k_t^\alpha n_t^{1-\alpha} - c_t - k_{t+1}]].
\]
The first order conditions are:

\[ c_t : \frac{1}{c_t} = \lambda_t; \quad (16) \]

\[ n_t : \frac{1}{1 - n_t} = \lambda_t (1 - \alpha) k_t^\alpha n_t^{-\alpha}; \quad (17) \]

and

\[ k_{t+1} : \lambda_t = \lambda_{t+1} \beta \alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}. \quad (18) \]

(16) and (18) imply that

\[ \frac{1}{c_t} = \frac{\beta \alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}}{c_{t+1}} \quad (19) \]

which is the Euler equation linking consumption in adjacent periods. To derive the path of consumption, capital stock, and employment we can use guess and verify method. Again assume that

\[ c_t = \mu k_t^\alpha n_t^{1-\alpha}. \quad (20) \]

Then using (14), (19) and (20) one can show that \( \mu = (1 - \alpha \beta) \) and

\[ c_t = (1 - \alpha \beta) k_t^\alpha n_t^{1-\alpha} \quad (21) \]

and

\[ k_{t+1} = \alpha \beta k_t^\alpha n_t^{1-\alpha}. \quad (22) \]

(16), (17), and (21) then imply that

\[ n_t = \frac{1 - \alpha}{2 - \alpha (1 + \beta)}. \quad (23) \]

In this model, agents supply same amount of labor every period. This is the consequence of logarithmic and separable utility function.

*Exercise: Derive steady state values of consumption, capital stock, and real wage.*
2. Business Cycle and The Real Business Cycle Model

So far we have considered models of perfect foresight. Now we extend our analysis to consider models with uncertainty. The incorporation of uncertainty will allow us to study business cycle issues. In particular, we are going to study the RBC model. This model has been very influential in studying business cycles. More generally, it has changed the methodology of macroeconomics and led to the emergence of Dynamic Stochastic General Equilibrium (DSGE) models. Most of the analysis in the modern macroeconomics is done using this framework. As discussed in lecture 2, the DSGE models have the following properties: (i) They specify budget constraints for households, technologies for firms, and resource constraints for the overall economy; (ii) They specify household preferences and firm objectives; (iii) They assume forward-looking behavior for firms and households; (iv) They include the shocks that firms and households face; (v) They are models of the entire economy.

Before developing the RBC model, we summarize the most important business cycle facts. Firstly, empirical evidence suggests that consumption, investment, and employment are highly pro-cyclical. Real wage is acyclic or mildly pro-cyclical. Secondly, the effect of temporary shocks on output is highly persistent. The effect of temporary shocks last up to eight quarters. Thirdly, the response of output to a shock is hump-shaped. The effect of shock on output peaks in fourth or fifth quarter. Fourthly, consumption is less volatile relative to output, while investment is highly volatile relative to output. Any good model of business cycle should be able to match and explain these facts.

Broadly, there are two approaches to study business cycles: (i) the RBC models and (ii) the Keynesian Models. There are two key differences between these two types of approaches. The RBC models assume Walrasian markets and they attribute business cycles primarily to technology (productivity/real/supply side) shocks. The Keynesian models on the other hand assume imperfect markets with nominal rigidities and they attribute business cycles primarily to aggregate demand or nominal shocks. In this lecture, we will develop RBC model and evaluate its performance in explaining business cycle regularities. The Keynesian models are covered in chapters 6 and 7 of Romer.
The Real Business Cycle Model

The RBC model is a stochastic version of the optimal consumption problem analyzed above. In the basic RBC model, it is assumed that there is uncertainty with regard to technology or production function. This model brings out the effects of technology shock on consumption, output, employment etc.

Consider a stochastic version of the optimal consumption problem analyzed above. Suppose that the production function is given by $A_t^\alpha$ where $A_t = \exp^{\epsilon_t}$. $\epsilon_t$ is an independently and identically distributed (i.i.d) random variable with mean 0 and variance $\sigma^2$. The random variable $A_t$ is called technology or productivity shock. It is assumed that the technology shock is realized at the beginning of period $t$ before consumption and investment decisions are made. Let $E_t$ denote the expectation operator conditional on time $t$ information set.

The optimization problem is to

$$\max_{c_t,k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t$$

subject to

$$k_{t+1} = A_t^{\alpha} - c_t.$$ (24)

This optimization problem is known as the real business cycle model, which studies the effects of technology shocks on investment, consumption, output etc. We can recast the above optimization problem as

$$\max_{c_t,k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \left[ \ln c_t + \lambda_t [A_t^{\alpha} - c_t - k_{t+1}] \right].$$ (25)

The first order conditions are

$$c_t : \frac{1}{c_t} = \lambda_t$$ (26)

$$k_{t+1} : \lambda_t = \alpha \beta E_t \lambda_{t+1} A_{t+1}^{\alpha-1}.$$ (27)

Combining (26) and (27) we get
\[
\frac{1}{c_t} = \alpha \beta E_t \frac{A_{t+1} k_{t+1}^{\alpha-1}}{c_{t+1}}
\]  
(28)

which is again the Euler equation. In order to solve (28), we will use the guess and verify method as before. Let

\[c_t = \mu A_t k_t^\alpha.\]  
(29)

Putting (29) in (28) we have

\[
\frac{1}{A_t k_t^\alpha} = \alpha \beta E_t \frac{1}{k_{t+1}}.
\]  
(30)

Combining (30) with the budget constraint we have

\[
\frac{1}{A_t k_t^\alpha} = \frac{\alpha \beta}{A_t k_t^\alpha - c_t}.
\]  
(31)

From (31) we get

\[c_t = (1 - \alpha \beta) A_t k_t^\alpha.\]  
(32)

(29) and (32) imply that \( \mu = 1 - \alpha \beta \). From (32) and the budget constraint we get

\[k_{t+1} = \alpha \beta A_t k_t^\alpha.\]  
(33)

Taking the log of (33), we get the fundamental stochastic difference equation which tells us how capital stock evolves over time.

\[\ln k_{t+1} = \ln \alpha \beta + \alpha \ln k_t + \epsilon_t.\]  
(34)

Using (34) we can trace out how capital accumulation evolves over time in response to a single shock, \( \epsilon_t \) (impulse response function). One can also derive moments of the process of capital accumulation. Solving (34) backwards we have

\[\ln k_{t+1} = \ln \alpha \beta + \alpha [\ln \alpha \beta + \alpha \ln k_{t-1} + \epsilon_{t-1}] + \epsilon_t.\]  
(35)

If we keep on repeating this process, we will get

\[\ln k_{t+1} = [1 + \alpha + \alpha^2 + \ldots] \ln \alpha \beta + [\epsilon_t + \alpha \epsilon_{t-1} + \ldots].\]  
(36)
Then

\[ E(\ln k_{t+1}) = \frac{\ln \alpha \beta}{1 - \alpha} \]  

(37)

\[ V(\ln k_{t+1}) = \frac{\sigma^2}{1 - \alpha^2}. \]  

(38)

Similarly we can derive moments of other variables like consumption, income etc. Denote output by \( y_t \). Then, we have

\[ \ln y_t = \ln A_t + \alpha \ln k_t. \]  

(39)

By combining (34) and (39) we have,

\[ \ln k_t = \ln \alpha \beta + \ln y_{t-1}. \]  

(40)

Then (39) and (40) imply that the log of output follows a first order autoregressive process:

\[ \ln y_{t+1} = \alpha \ln \alpha \beta + \alpha \ln y_t + \xi_{t+1}. \]  

(41)

Let us now consider implications of technology shock on consumption, investment, and output. Using (41) one can derive the implications of technology shock on output. For this one needs to assume some value of \( \alpha \). \( \alpha \) is estimated to be 0.33.

Now consider the effect of a one time positive technology shock. Let \( y_0 \) be the output at time \( t - 1 \). Suppose that at time \( T \) technology shock is realized and let \( \xi_T = 1 \). Suppose that the technology shock is entirely temporary, i.e. \( \xi_t = 0, \forall t > T \). What would be the response of output? In period \( T \), the log of output will be one unit higher than the log of \( y_0 \). In \( T + 1 \), it will be higher by \( 1/3 \) compared to the log of \( y_0 \). In \( T + 2 \), it will be higher by \( 1/9 \) and so on.

There are two problems with this kind of response. First, the response of output is not hump shaped as in the data. Output rises in period \( T \) and then falls linearly. Secondly, the effect of technology shock is not persistent. Its effect dies down quite quickly. There are other problems as well. In the model, consumption and investment are a constant proportion of output. Thus, both are as volatile as output. However, in the real data consumption is much less volatile than output and investment is much more volatile. This
analysis suggests that this version of RBC model does not do a good job in accounting for business cycle facts.

Now let us modify the environment as follows. Suppose now that technology shock follows an autoregressive process:

\[ \xi_{t+1} = \rho \xi_t + u_{t+1} \]  

(42)

where \( u_t \) is i.i.d. with mean zero and variance \( \sigma^2 \). (41) can be written as

\[ \ln y_{t+1} = \alpha \ln \alpha \beta + \alpha \ln y_t + \rho \xi_t + u_{t+1}. \]  

(43)

(41) also implies that

\[ \xi_t = \ln y_t - \alpha \ln \alpha \beta - \alpha \ln y_{t-1}. \]  

(44)

Then (43) and (44) imply that the log of output follows a second order autoregressive process:

\[ \ln y_{t+1} = \text{Constant} + (\alpha + \rho) \ln y_t - \alpha \rho \ln y_{t-1} + u_{t+1}. \]  

(45)

(45) shows that the response of output to technology shock depends on two parameters, \( \alpha \) and \( \rho \). If \( \rho \) is high, meaning that technology shock is highly persistent, the model can generate a hump-shaped and persistent response of output to technology shock. However, it does not fix other problems.

The models considered so far do not say anything about employment and real wage. To consider the implications with regard to employment and real wage, let us introduce technology shock in example 2. Now the problem is

\[ \max_{c_t, n_t, k_{t+1}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \left[ \ln c_t + \ln (1 - n_t) + \lambda_t \left[ k_t^{\alpha} n_t^{1-\alpha} - c_t - k_{t+1} \right] \right] \right]. \]  

(46)

The first order condition for consumption continues to be given by (16). The first order conditions for employment and capital stock modifies to

\[ n_t : \frac{1}{1 - n_t} = \lambda_t (1 - \alpha) A_t k_t^{\alpha} n_t^{1-\alpha}; \]  

(47)

and

\[ k_{t+1} : \lambda_t = \beta \alpha E_t \lambda_{t+1} A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha}. \]  

(48)
(16) and (47) imply that

\[
\frac{1}{c_t} = \alpha \beta E_t A_{t+1} k_{t+1}^{\alpha} n_{t+1}^{1-\alpha} c_{t+1},
\]

which is the Euler equation linking consumption in adjacent periods. Using the guess and verify method, one can show that

\[
c_t = (1 - \alpha \beta) A_t k_t^{\alpha} n_t^{1-\alpha};
\]

\[
k_{t+1} = \alpha \beta A_t k_t^{\alpha} n_t^{1-\alpha}
\]

and

\[
n_t = \frac{1 - \alpha}{2 - \alpha(1 + \beta)}.
\]

Since \( n_t \) is constant, the model implies that technology shock does not affect employment. Also since real wage is equal to the marginal product of labor \( \left( = (1 - \alpha) y_t / n_t \right) \), real wage is strongly pro-cyclical. Both these predictions are at odds with empirical evidence. As discussed earlier, employment is highly pro-cyclical and real wage is essentially acyclical.

What we have seen that the basic RBC models cannot account for most of the business cycle facts. Researchers have modified basic RBC models in many ways to improve the performance of these models. Some of the most prominent extensions are introduction of government expenditure and fiscal shocks, taste shocks (shocks to utility function), more complicated utility function (habit formation), shocks to inventory etc. Many researchers have taken more fundamental departures abandoning the assumptions of competitive or frictionless markets. Now, it is customary to assume frictions in either the labor market or the goods market or both. Frictions in the labor market generate unemployment and have allowed researchers to analyze unemployment dynamics. Some models also incorporate credit markets. These extensions have improved the performance of these models.