1. **Richardson Number Stability Criterion for Continuously Stratified Shear Flows**

Consider a stratified Boussinesq flow with mean background zonal wind $U(z)$ and potential temperature $\theta_0(z)$. Suppose further that there is no mean meridional flow and that the flow is invariant for translations along $y$. Denote by $u, w, t, \theta$ and $p$ small perturbations to the zonal wind, vertical wind, potential temperature, and pressure, respectively. The linearised equations of motion for the perturbation quantities are (ignoring rotation):

\[
\begin{align*}
(\partial_t + U\partial_x)u + w\partial_z U &= -\frac{1}{\rho_0} \partial_z p \\
(\partial_t + U\partial_z)w &= -\frac{1}{\rho_0} \partial_x p + g \frac{\theta}{\theta_0} \\
\partial_x u + \partial_z w &= 0 \\
(\partial_t + U\partial_x)\theta + w\partial_z \theta &= 0 
\end{align*}
\]  

(a) Show that these equations can be combined to produce an equation for $w$ alone:

\[
\partial_{xx}w + \partial_{zz}w + 2U\left(\partial_{xx}w + \partial_{zz}w\right) + U^2\left(\partial_{xxxx}w + \partial_{zzzz}w\right) - \partial_{zz}U\left(\partial_{zt}w + \partial_{xz}w\right) + N^2\partial_{xx}w = 0 
\]  

where

\[
N^2 = \frac{g}{\theta_0} \frac{d\theta_0}{dz} 
\]

is the Brunt-Vaisala frequency of the background stratification.

(b) Assuming a plane wave form for $w$:

\[
w = \psi(z) \exp(ik(x - ct))
\]

show that

\[
(U - c)^2 \frac{d^2 \psi}{dz^2} + \left[ N^2 - (U - c) \frac{d^2 U}{dz^2} - k^2(U - c)^2 \right] \psi = 0 
\]

This is the **Taylor-Goldstein Equation**.

(c) Defining

\[
\phi = \frac{\psi}{\sqrt{U - c}} 
\]

show that the Taylor-Goldstein equation (A5.8) can be written

\[
\frac{d}{dz} \left[ (U - c) \frac{d\phi}{dz} \right] - \left[ k^2(U - c) + \frac{1}{2} \frac{d^2 U}{dz^2} + \frac{1}{U - c} \left( \frac{1}{4} \left( \frac{dU}{dz} \right)^2 - N^2 \right) \right] \phi = 0 
\]

(d) Assume that the perturbation vertical velocity vanishes at $z = z_1, z_2$, so that

\[
\phi(z_1) = \phi(z_2) = 0 
\]

Multiplying equation (A5.10) by $\phi^*$ (where * denotes complex conjugate) and integrating the resulting equation over $z$, show that

\[
\int_{z_1}^{z_2} \left( N^2 - \frac{1}{4} \left( \frac{dU}{dz} \right)^2 \right) |\phi|^2 d\frac{dz}{(U - c)} = \int_{z_1}^{z_2} \left( \frac{d\phi}{dz} \right)^2 + k^2|\phi|^2 dz + \frac{1}{2} \int_{z_1}^{z_2} \frac{d^2 U}{dz^2} |\phi|^2 dz 
\]  

1
(e) In general, a wave speed \( c \) satisfying equation (A5.12) will be complex:

\[
    c = c_r + ic_i \tag{A5.13}
\]

Show that if \( c \) satisfies (A5.12), so does its complex conjugate \( c^* \) (note that \( k \) is defined to be real). Thus, if there is a solution \( c \) to A5.12 with a positive imaginary part, there must also be a solution with an negative imaginary part. In other words, if \( c_i \neq 0 \), there will be an exponentially growing solution to (A5.12) and the perturbation will be unstable.

(f) Show that the imaginary part of equation (A5.12) can be written:

\[
    c_i \int_{z_1}^{z_2} \left[ \left( N^2 - \frac{1}{4} \left( \frac{dU}{dz} \right)^2 \right) \left( |\phi|^2 + \left| \frac{d\phi}{dz} \right|^2 + k^2 |\phi|^2 \right) \right] dz = 0 \tag{A5.14}
\]

Show furthermore that the flow will be stable if

\[
    Ri > \frac{1}{4} \tag{A5.15}
\]

everywhere in the flow, where

\[
    Ri = \frac{N^2}{(dU/dz)^2} \tag{A5.16}
\]

is the gradient Richardson number. Finally, show that instability is possible if \( Ri < 1/4 \) in some part of the flow.

2. Baroclinic Instability of a Continuously Stratified Atmosphere (Eady Model)

Consider a continuously stratified atmosphere on a midlatitude f-plane \((\beta = 0)\) with background wind profile (in log pressure coordinates)

\[
    U(z^*) = \Lambda z^* \tag{A5.17}
\]

where \( \Lambda \) is a constant. For simplicity, assume that the density \( \rho_0 \) appearing in the log-pressure coordinate expression for quasigeostrophic potential vorticity (A3.33) is constant, as is the stability parameter \( \varepsilon \), so that

\[
    q = \nabla^2 \psi + f + \varepsilon \partial_{z^*z^*} \psi \tag{A5.18}
\]

(a) Given the above approximations, show that the equation of conservation of quasigeostrophic potential vorticity linearised around the wind \( U(z^*) \) is

\[
    (\partial_t + U \partial_x) (\nabla^2 \psi' + \varepsilon \partial_{z^*z^*} \psi') = 0 \tag{A5.19}
\]

where \( \psi' \) is the perturbation streamfunction. Assuming further a wave form for \( \psi' \):

\[
    \psi'(x, y, z^*, t) = \Psi(z^*) \cos(ly) \exp(ik(x - ct)) \tag{A5.20}
\]

show that

\[
    \frac{d^2 \Psi}{dz^*2} - \alpha^2 \Psi = 0 \tag{A5.21}
\]

where

\[
    \alpha^2 = (k^2 + l^2)/\varepsilon \tag{A5.22}
\]

(b) For boundary conditions, assume rigid upper and lower boundaries so that the perturbation vertical velocity vanishes at \( z^* = 0 \) and \( z^* = D \) (the surface and the tropopause, respectively). To use these boundary conditions, we need to relate them to \( \psi' \). Combine the adiabatic thermodynamic energy equation linearised around \( U(z^*) \) with these boundary conditions to show that

\[
    (\partial_t + U \partial_x) \partial_{z^*} \psi' - (\partial_{z^*} U) \partial_x \psi' = 0 \tag{A5.23}
\]

at \( z^* = 0, D \). Finally, use (A5.20) to show that

\[
    (\Lambda z^* - c) \frac{d\Psi}{dz^*} - \Lambda \Psi = 0 \tag{A5.24}
\]

at \( z^* = 0, D \).
(c) A general solution $\Psi$ to equation (A5.21) can be expressed

$$\Psi = A \sinh \alpha z^* + B \cosh \alpha z^*$$  \hspace{1cm} (A5.25)

Show that the boundary conditions imply the pair of simultaneous linear equations:

$$-c\alpha A - \Lambda B = 0$$  \hspace{1cm} (A5.26)

$$(\alpha(\Lambda D - c) \cosh \alpha D - \Lambda \sinh \alpha D) A$$

$$+ (\alpha(\Lambda D - c) \sinh \alpha D - \Lambda \cosh \alpha D) B = 0$$  \hspace{1cm} (A5.27)

and that these equations have a nontrivial solution only if

$$c = \frac{\Lambda D}{2} \left\{ 1 \pm \frac{2}{\alpha D} \left[ \left( \frac{\alpha D}{2} - \coth \frac{\alpha D}{2} \right) \left( \frac{\alpha D}{2} - \tanh \frac{\alpha D}{2} \right) \right]^{1/2} \right\}$$  \hspace{1cm} (A5.29)

(d) Clearly, the system will be baroclinically unstable if the quantity in square brackets in equation (A5.29) is negative. Show that instability occurs if (and only if)

$$\alpha < \alpha_c$$  \hspace{1cm} (A5.30)

where $\alpha_c$ satisfies:

$$\frac{\alpha_c D}{2} = \coth \frac{\alpha_c D}{2}$$  \hspace{1cm} (A5.31)

Show that

$$\alpha_c \approx \frac{2.4}{D}$$  \hspace{1cm} (A5.32)

(this can’t be done analytically, so you can do it graphically). Note that the instability criterion implies that

$$(k^2 + l^2) < 5.76 \kappa^{-2}$$  \hspace{1cm} (A5.33)

where

$$r = ND/f_0$$  \hspace{1cm} (A5.34)

is the Rossby radius of deformation for the stratified fluid.

(e) When the flow is unstable, how does the speed of the growing wave compare with the speed of the wind in the lower troposphere? What about in the upper troposphere?

(f) Considering the case in which $k = l$ (with $k > 0$), show that the growth rate $kc_i$ can be written

$$\left( \sqrt{\frac{2}{\varepsilon}} \frac{1}{\Lambda} \right) kc_i = \left[ \left( \coth \frac{kD}{\sqrt{2\varepsilon}} - \frac{kD}{\sqrt{2\varepsilon}} \right) \left( \frac{kD}{\sqrt{2\varepsilon}} - \tanh \frac{kD}{\sqrt{2\varepsilon}} \right) \right]^{1/2}$$  \hspace{1cm} (A5.35)

Plot the growth rate as a function of $k$ over the domain where it is positive. From this plot, determine the value of $k$ which is most unstable (i.e., has highest growth rate). Contour the perturbation streamfunction and temperature as functions of $x$ and $z$ for the most unstable wave. How do lines of constant streamfunction phase tilt with altitude? What is the dynamical significance of this tilt?