1st Order Ode Flowchart & Recipe Book

To solve the ODE
$$\frac{dy}{dx} = f(x, y)$$
it is useful to ask yourself the following:

1. **Is the ODE separable?**

   Can the ODE be written
   $$\frac{dy}{dx} = g(x)h(y)$$
   for some functions $g(x)$ and $h(y)$? If so, solution is found from:
   $$\int \frac{dy}{h(y)} = \int g(x)dx + C$$

   IF NOT,

2. **Is the ODE linear?**

   Can the ODE be written
   $$\frac{dy}{dx} + P(x)y = f(x)$$
   for some functions $P(x)$ and $f(x)$? If so, then solution is found from
   $$q(x)y(x) = \int q(x)f(x)\; dx + c$$
   where
   $$q(x) = \exp\left(\int P(x)dx\right)$$

   IF NOT,

3. **Is the ODE exact?**

   Can the ODE be written
   $$M(x, y)dx + N(x, y)dy = 0$$
   for some functions $M(x, y)$ and $N(x, y)$ such that
   $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
   If so, then have the implicit solution $h(x, y) = const.$ where
   $$h(x, y) = \int M(x, y)\; dx + g(y)$$
   and the function $g(y)$ is found from requiring that
   $$\frac{\partial h}{\partial y} = N(x, y)$$

   IF NOT,

4. **Can we solve the ODE by substitution?**

   (a) *Is the ODE a homogeneous equation of order α?*
   Can we write the ODE as

$$M(x, y)dx + N(x, y)dy = 0$$

   for some functions $M(x, y)$ and $N(x, y)$ such that $M(tx, ty) = t^\alpha M(x, y)$ and $N(tx, ty) = t^\alpha N(x, y)$ for some constant α? If so, then with the substitution $y = ux$ (such that $dy = udx + xdu$), the ODE becomes separable in u and x.
   IF NOT

   (b) *Is the ODE of the Bernoulli Type?*
   Can we write the ODE in the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

   If so, then the substitution $u = y^{1-n}$ makes the equation linear in u.
   IF NOT

   (c) *Can we find some other substitution?*
   The answer to this is down to (educated) guesswork.

IF NOT

5. **Can we describe the qualitative solution to the ODE?**

Is the ODE autonomous, that is, of the form

$$\frac{dy}{dx} = f(y)$$

If so, we can find the critical points, that is, solutions of the equation $f(y) = 0$. The critical points can be classified as attractors or repellers, and the basins of attraction of the attractors found. This allows the qualitative asymptotic ($x \to \infty$) behaviour of the solutions to be characterised.

OTHERWISE

6. **Can we use a numerical method?**

If qualitative solutions are not sufficient, or not available, then we may simply have to recourse to obtaining numerical approximations.