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Incarnation: Radicalizing the Embodiment of Mathematics

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As the end result of metaphysics, Kant's constructive mind is not present in the world but withdrawn into the netherworld of its representations and constructions. First phenomenology then embodied cognition research suggested that there could be no cognition without the human body. There is something unsatisfying and lacking, however, in the concept of the *body*, which undermines the very effort to ground (mathematical) knowledge differently than in the private cogitations of the isolated mind.[1] The purpose of this paper is to articulate a conceptualization of mathematical knowledge that is grounded in materialist phenomenology (as developed by Maine de Biran and Henry) and phenomenological sociology (Bourdieu). Most essentially, the approach rests on a shift from the category of the (material) *body* to that of the *flesh*. The close relation between cognition and the world is possible only when there is flesh, whereas the material body is an insufficient condition for the *human* mind to emerge. In other words, current embodiment theories tend not to answer questions about the special characteristics of those bodies that develop human minds versus those bodies that do not and cannot. In this phenomenological reflection on the conditions of geometrical

[1] In this article, space limitations prohibit a critique of constructivism or the going scholarship on embodiment, as any attempt could only do injustice to these impressive bodies of work. Full, chapter-length critiques of each approach are included in my forthcoming *Mathematics in the Flesh*. In the chapters, I confront each of these theories with fragments from geometry lessons to show what they cannot explain and why.

knowing, I draw on a lesson fragment from a second-grade classroom to exemplify this different conceptualization of mathematical knowing.

How does mind make contact with the world?

> I stepped straightly into phenomenology, which was a bit didactically tainted.
>
> (Freudenthal, 1983, p. 210)

Even the first tasks that allow children to engage with three-dimensional shapes allow the objective nature of geometry to emerge from the events in an elementary classroom (Roth& Thom, 2009). The purpose of this paper is to offer a different conceptualization for thinking the "embodiment" of mathematical knowledge. To get the point of this article you have to begin by enacting the following task. Through this exploration, I intend us to focus on the phenomenology of geometrical and spatial experience – which differs in cultures other then our own (e.g., Pinxten, 1991). As for Freudenthal, my stepping into phenomenology – though mine is different than his – is "a bit didactically tainted."

Take what you have learned to be a cube in your hand. Look at the object. Do you see a cube? . . . Of course not: you see aspects (parts) of a cube depending on your current perspective and the orientation of the object. But how do you know that these are aspects of a *cube*? To know this, you already need to know that a cube expresses itself in the aspects you see given perspective and orientation. Close your eyes and feel the object. Do you feel a *cube*? . . . Of course not. Even if the object is small so that you can completely enclose it with your hands, you do not feel the twelvefoldedness of the edges, the sixfoldedness of the sides, and the eightfoldedness of the corners, the smoothness/roughness of the surfaces, and so on. Especially, you do not feel all of these aspects of a *cube* simultaneously. Now, consider this: following my instructions, you intended to explore the cube by means

of sight and touch.[2] But where did this intention come from? Did you have an intention to have the intention of exploring the cube?

Some time in our personal histories, the *cube* has arrived in our consciousness, as a result of our lived experiences. But before that time, the object, the cube as cube, "does not give itself so long as its signification is not adequately fulfilled . . . The paradigm for this impossibility is seen in even the most simple intentional object (the cube . . .), whose geometric faces we never, by an essential impossibility, perceive all at once" (Marion, 2002, p. 126).[3] How is it that our understanding of the cube as a cube has arrived even though our mind did not have the concept of a cube to see and recognize one? How, without a "plan" of what a cube is or looks like, does a constructive mind arrive at a cube from the disparate sensual (visual, tactile) experiences that a learner may have with the object that we now know to be a cube? How can the mind intend constructing a cube when it does not know what a cube is and therefore cannot intentionally aim at and construct it? This is the dilemma that Kant, the "logodedalus" (Nancy, 2008), the master artisan of the mind, has never solved. But then how did we come to understand that by looking at or touching some solid square faces we were seeing / touching a cube?[4]

The problem for the intellectualist mind is known in the cognitive sciences as the "symbol grounding problem," that is, the problem how mental representations are related to anything other than mental representations (Harnad, 1990). How does mind know that what it has constructed relates to anything in the world? Because social constructivists (e.g., Cobb, 1999) cannot answer this question, they have to

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[2] In this paper, I only deal with touch (due to space). However, philosophical arguments from Aristotle to the present day (e.g., Derrida, 2005) exhibit both the primacy of touch to cognition and its paradigmatic function in understanding all other senses.

[3] All translations from French are mine.

[4] Here, I am not going down the discursive road of and surrounding "meaning," because it is part of the problem when knowing is reduced to words, when in fact part of knowing is not verbal at all. We denote aspects of knowledge by means of a term that is precisely not of the kind.

assume everything to be "taken-as-shared." How does perception, given its theory-laden nature, ever come to see a cube when any possible experience exhibits only perspectives that never reveal the totality of things we know today about a cube (8 corners, 12 sides, 6 square faces intersecting at 90 degrees, etc.)? The general constructivist answer is that only fit and usefulness matter rather than real relations. But then the question is how mind can interact with the physical world, how the abstract mind can manipulate the body and the senses and test its knowledge in the world? In fact, it is impossible to understand how an intellectualist mind could do anything like paint in the outside world (Merleau-Ponty, 1964b). Moreover, how can the knowing subject know itself if the known cannot be anything other than an ob-ject, something thrown (Lat. *iacere*, to throw) in front of, before, against (Lat. *ob*-) the subject? In Kant's constructivist approach, the knowing subject and the object known are but two abstractions and a real, positive connection between the two does not exist (Maine de Biran, 1859a, b). The separation between inside and outside, the mind and body, is inherent in the intellectualist approach whatever the particular brand. This is why Ernst von Glasersfeld's (e.g., 1989) radical constructivist mind knows nothing about the world and remains stuck in its own representations that are tested for fit.

The past decade and a half has seen a slow increase in the number of studies that argue for an embodiment approach to mathematical knowing and learning (e.g., see the special issues on gesture, *Educational Studies in Mathematics, 70* (2), 2009, and on bodily activity and imagination, *Educational Studies in Mathematics, 57* (3), 2004). Some mathematicians and mathematics educators, however, resist the idea, holding fast to the conceptualization of knowing in terms of mental representations. In part, this resistance is understandable, as the current embodiment literature concerning mathematical knowing and learning may fall short on three accounts (discussions with Luis Radford, February 24, 2010). First, it may not provide a

sufficient account of how bodily knowledge comes to be the *necessary* condition of mathematical understanding rather than merely being a stepping-stone that can be kicked aside once real mathematical knowledge has emerged through metaphorical and metonymic *mapping procedures* into the mental domain (e.g., Lakoff & Núñez, 2000; Varela, Thompson & Rosch, 1991).[5] Second, this literature may not sufficiently explain how metaphorical and metonymic *mapping* can come about given that this process requires the very linguistic and language-like mathematical knowledge that is to be explained. What and wherefrom is the [mathematical] domain *y* onto which bodily knowledge *x* comes to be mapped by the function f (i.e., f: x → y)?[6] Third, the current embodiment literature may not sufficiently articulate how the collective body of knowledge, the *corpus* (body) of objective mathematics, is a condition for individually knowledge grounded in the body (corpus).

The purpose of this paper is to articulate an alternative to standard embodiment approaches. The standard approaches to embodiment do not make the *categorical* distinction widely discussed within phenomenology, the one between the body and the flesh. I suggest taking the flesh rather than the body as the ground of all knowing: knowledge as incarnated, enfleshed. Because it is enfleshed, mathematical knowledge also is embodied. It is the flesh where we find tact (touch), contact, and contingency, and therefore, the ground of knowledge so that the sense of the body comes to be the body of sense (Nancy, 2006). To anchor the discussion, I begin with a fragment extracted from a videotape recorded in a second-grade classroom in which teacher and students have embarked to study geometry. This fragment has an exemplary function and has been selected on pragmatic grounds from a large

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[5] "The sensualist or materialist reduction and idealist teleology follow the same line, in inverse sense. The one we just called . . . 'metaphor'" (Derrida, 1972, p. 105).

[6] For a dialectical critique of metaphors, including the one of metaphor itself see Derrida, 1972, especially the section entitled "Plus de métaphore" (pp. 261–273), an untranslatable title (the official English translation leaves the title in French), as it means both "more metaphor" and "no more metaphor."

number of structurally similar instances in my data sources. I then articulate a conceptualization of embodiment, or incarnation, grounded in the work of a little-known 18th and 19th century French philosopher Maine de Biran, whose work has found prominence more recently in *material phenomenology* (e.g., Henry, 1965, 2000). A materialist phenomenological approach theorizes knowing beginning with very primitive forms of experiences that precede mind and intention.

"What makes it a rectangular prism instead of a cube?"

The fragment below begins after a description that the second-grader Chris provided about rectangular prisms that can have different lengths of the sides. The teacher of his mathematics class, which has embarked on an exploration of three-dimensional geometry, first summarizes what she has heard Chris to say, and then follows up asking about what the pizza box would have to have to make it a cube (turn 17). At this time, Chris stands at the front of the classroom where the pizza box rests next to the chalkboard; he is slightly bent over the box (Figure 1a). For much of the time in the lesson fragment, Chris is oriented toward the pizza box and therefore has his back facing his classmates. It is only near the end of the fragment, when he points to the familiar cube in his hand, that he reorients to the teacher and gazes at her face (Figure 1b), producing a more articulate utterance concerning the nature of a cube. Initially, his verbal production may be described as inarticulate, but full with marks of the temporal, embodied production of communication (e.g., gestures, body orientations, body position, intonations, changes in delivery speed, changes in speech volume, pauses). This other aspects of communication hardly ever play any role in the analyses among embodiment advocates, even though these are integral to and constitutive of the expression as a whole and the way in which

speakers position themselves with respect to topic and others. But this initial part is even more important in and to my case than the second part.

«««««« Insert Figure 1 about here »»»»»»

Articulating an answer and the world: an episode

After the teacher's utterance, there is a pause (turn 18). The silence is broken by an almost simultaneous beginning of each of the two speakers (turns 19 & 20), with the teacher speaking quickly to complete an utterance that translates the preceding question: "what would that box have to have to be a cube" (turn 20). There is another pause before Chris begins using some of the teacher's words, "it would have to" and then produces a interjection preceding a pause (turn 22). The production is punctuated by pauses of different length, "that turn," "or this," "like the," "sorta like" and then another long pause of 3.04 seconds. The verbal part of Chris' presentation continues in a stop-and-go manner punctuated by pauses, "square here an like" and then his voice fades away becoming all but inaudible "but to-n-this-squ." Transcribed, his articulation so far has been, "um this . . . it would have to um that turn or this like these . . . sorta like . . . um square here an like." He then turns to the teacher until squarely facing her and produces a rapid utterance, "it has square here and here and here and everywhere."

Lesson Fragment[7]

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17  T: so <<f>that> makes it a rectangular prism as opposed to a cUBe,
       because if it was a cube what would it have to have
18     (0.58)
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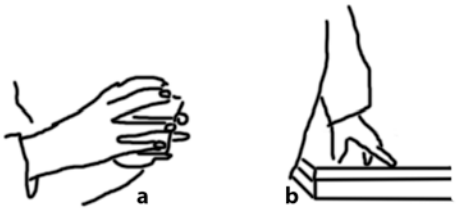
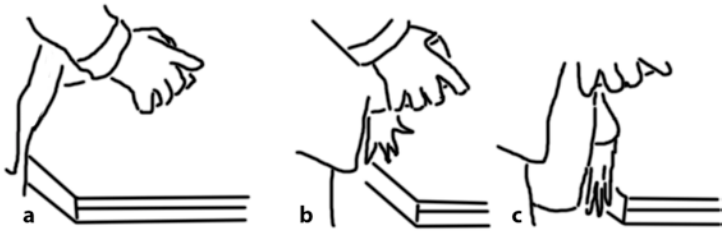
[7] The following transcription conventions have been used: The transcription reproduces sounds, not grammar or capitalization; colon (:) indicates lengthening of sound by one tenth of a second; comma and period indicate slightly rising and strongly falling pitch toward the end of the utterance, apostrophe (') indicates rising pitch; speaking pauses are indicated in hundredth of seconds (0.77); speech features are indicated in triangular brackets, <f> (forte) referring to louder, <all> (allegro) to faster, <pp> (pianissimo), much more quiet than normal speech; equal sign means latching of sounds; and double parentheses mark off transcriber's comments.

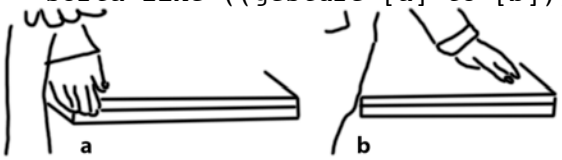
19 C: um [thiss]
20 T: <<all>[what] would> that box have to have to be a cube.
21 (0.77)
22a C: it would have to um ((gets a cube from chalk holder)) (0.44) that tu:rn: ((a)) or this (0.21) like the:(0.62)s:: ((b))

22b 

22c ((bends down to box, moves cube in direction of box, moves left hand along edge of pizza box))



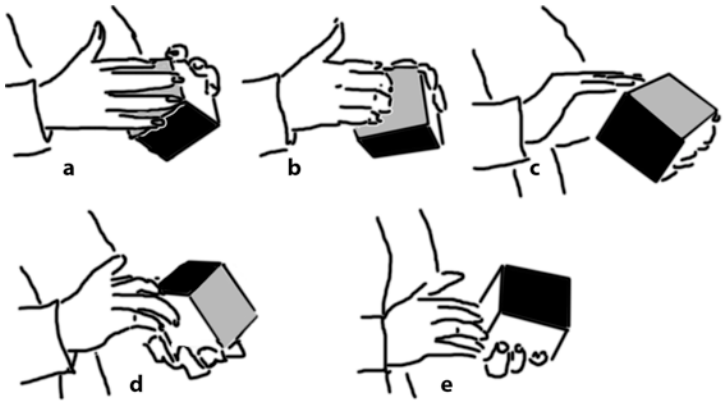
22d sorta like ((gesture [a] to [b]))



22e (3.04) ((hand moving along edge)) um (0.23) s::quare

22f here

22g an like ((rectangle)) (0.68) <<pp>but to=n=this squa> ((movement along edge of box)) it just has (([a])) square (([b])) (0.45) here= (([b])) =an=here= (([c])) =an here= (([d])) =and everywhere (([e]))



23 (0.63)
24 M: ‘how could you make a cube from pizza boxes.

In the early part of the response, Chris is entirely oriented to the pizza box (Figure 1a). In this, a relation establishes itself between the box and him, from which is to emerge a response. His orientation exhibits the nature of the preceding utterance as a question, which he shows for everyone to see as the constitution of the second part of the turn pair that establishes the question–answer as a unit. This orientation also communicates something like absorption, of the subject (Chris) with its object. Here the object is not the pizza box in itself but the question about this entity and what would make it a cube. The initial part of the “response” (turn 22) is produced together with an inarticulate production in the verbal mode. Chris moves his hand along the two edges / sides of the pizza box that are visible to the camera and closer to him (turn 22c, gesture b → c; turn 22d, gesture a → b). There is a very long pause, and he then utters the word “square” (turn 22e) while he points to the two sides again (turns 22e, f). Seen and heard in response to the question “what would that box have to have to be a cube,” these communicative productions appear to indicate the two sides that have to be “sorta like square ((points to one rectangular side of box)) here ((points to other rectangular side of box)).” But whereas we can hear / see what he articulates and points out, we do not know what is in his mind, what representation, mental picture, or conception he might have. In

fact, all of this is irrelevant in the face of the fact that he can be witnessed to be responding appropriately.

Senses and sense making

Intellectualist approaches reduce expressions of knowing to mental representations and therefore to an *external* relation of the subject to the object (Bourdieu, 1997). But the hand movement that we see in the transcribed fragment is not subject to the same mode, the verbal; it is to be thought differently, as a production that the student consciously controls. We should not hastily assume that underlying this movement there is anything similar to an adult thought, a conception, which has a hard time expressing itself in words. We cannot even assume that Chris has heard the question in the way we hear (read) it. We also know that the pizza box and later the cube appear in the same form of the present, but we do not know whether this presence is present to Chris, whether and how these objects are present to his consciousness, which would require re-presentation. But how then should we understand the hand movement? How come Chris does move in the way he does, sliding his hand along the edges, pointing, while uttering words that do not constitute a coherent and "logical" argument?

The response to this question takes us to an instant that even precedes the "bodies in space" as the phenomenological primitives on which geometrical understanding is built (Freudenthal, 1983). There is knowing (-how) and memory in the body, which can only come from an originary "I can," the power to act in this way. Chris's hands are able to move and he can intend moving this way because the sensorimotor mechanism remembers that it can move, even prior to any conception and intention (Maine de Biran, 1841). Intention, therefore, is not just "a property of human mental states" (Johnson, 1987, p. 181), but it is the emergent result of an auto-affecting flesh. Chris's hand and fingers glide along edges, across the box, and

along another edge. And to proceed in this manner, his body needs to know – *prior* to all intention, *prior* to any mental states – that such movements are within its range of possibilities. It is impossible to intend moving the hand unless the hand already knows how produce this movement; intention is the *result* of a capacity to move not the genetic origin of the movement. Chris gazes, but we do not know what he is consciously seeing and which aspects constitute a part of the guiding of movement in the unity of perception movement.

In fact, the fragment points us to a time before words, a time "before" time, before intentionality, when in the first hand-arm movements, Chris, before he was conscious of being Chris, realizes the possibility to move the hands voluntarily to reach out and touch. From Aristotle to the present day, scholars have recognized the primacy of touch in human experience: "no living being in the world can survive for an instant without touching, which is to say, without being touched" (Derrida, 2005, p. 140). Whereas it is possible to live without seeing, hearing, tasting, and smelling, "we cannot survive one instant without being with contact, and in contact" (p. 140). The sensations from the eyes and other external organs come to be coordinated with the sensations of touch, especially with touch from the hands. But this coordination is not the result of a constructive mind, who would not know that the input from the hands and eyes refer to the same thing in the world. It is a more originary "I can" in which this coordination comes about: "The thought of the consciousness-pole is born with the manifestation that it renders visible without knowing or wanting it, and perhaps without being able to do so" (Marion, 2002, p. 265). And with the movements and coordination of movements of eyes and hands, the world begins to emerge from touch. Chris's present experience is based on the coordination of hands with eyes, so that seeing the pizza box and moving the hand along one edge, then another edge, is but a realization of the coordination of hands and eyes and the concrete realization of the ability of moving them.

Certainly of equal if not greater importance to the radical approach to embodiment outlined here is the second part of Chris's explanation, in which the rectangular prism of the pizza box comes to be set against the cube ("but to-n-this . . ."). But Chris does not just contrast the rectangular prism (pizza box) with the cube or state some properties that one has but the other does not. Rather, uttering nothing more than "it has square here," and, as he rotates the cubical object touching different surfaces, says "here" each time, only to end in the totalizing statement "and everywhere."

The visible and the invisible mind

Relating to the phenomenology of perception, philosophers have noted that knowing a cube means knowing what it will look like when the relative orientation between subject and object changes (e.g., Merleau-Ponty, 1945). Here, although Chris does not produce a formal definition of the cube, he enacts the requisite experience for knowing what a cube is. He rotates the cube, exposing its different, initially hidden faces so that he comes to look at them "squarely," allowing him to see the square. Moreover, he says "it just has square," and then his left hand turns the cube and his right hand points while uttering "here . . . and here . . . and here and everywhere" (turn 22g). Now it would be wrong if we were to say that he has a conception of the cube and assume that this conception exists in verbal form; even less must we assume that he has some formal conception or knowledge a priori from which he could derive the properties of a cube, knowledge that is there but invisible. At this stage, what he makes available to us is the experiential fact that the object that goes with the sound transcribed as "kju:b" (in the conventions of the International Phonetics Association) can be turned to exhibit squares "everywhere." Moreover, we do not know about an explicit intention to rotate the object, but, seeing the object rotate in his hands, we know of his hands' ability to make the cube

rotate and exhibit its different faces. Chris makes available to us a description of what he sees when he rotates the object in his hands: Squares everywhere.

We often hear mathematics teachers and mathematics educators speak about students who know but who cannot articulate their knowing – how could these educators know without there being *some* expression of this knowing? In the research on gestures, there are indications that students may have one (McNeill, 2002) or two underlying conceptions (Alibali & Goldin-Meadow, 1993) that express themselves in their hand gestures and that may (or may not) be accompanied by words that express the same or less advanced conceptual content. Gesticulation is unconsciously produced (McNeill, 1992), so that this aspect of body movement cannot be theorized as falling under consciousness and cognition. The point of the description in the previous paragraph is precisely that: Embodied cognition cannot mean that the body, too, is subjected to the reign of intellectualism. A phenomenologically and ethnographically adequate description must take into account what is actually there, not what is imputed to the person by the theoretical presuppositions that underpin an intellectualist view of the mind. Even talk about the body is insufficient, because the cube and the pizza box also are material bodies, and we need to distinguish them, incapable of cognition, from Chris, exhibiting cognition in his description. Rather than thinking about the material body as the ground of cognition, therefore, we need to think about and theorize the flesh, which is the source of the body in cognition. Without the flesh, there cannot be knowledge of material bodies, including our own. This problem exists particularly if researchers tell us "(word, verbal) meanings" behind the gestures, some core idea, which is the very point we need to question.

Toward incarnation

Rather than theorizing gestures and body movements in terms of hidden knowledge or hidden conceptions, which conflates and confuses explicit and implicit forms – i.e., knowing what and knowing how – I propose shifting our theoretical discourse toward that which is immanent. Gesticulations are unconscious, and it is the unconscious production that I focus on in this episode. The production of gesticulation goes alongside with an inarticulate verbal presentation (i.e., making present). But to say that he has an "understanding" taken in the common sense of "to comprehend by knowing the sense of words used" would be an overstatement, precisely because of the absence of words. Something else is immanent in Chris's movements. In the present situation, what is immanent in his hand and finger movement precisely is the memory of the movements he can produce, and neither memory nor the production requires verbal thought. There is an "I can" immanent to the power of the arm/hand/finger to move in this way; and this "I can," prior to any "I," is the result of an auto-affection by means of which the muscles become empowered and know how to move. Auto-affection, one that operates prior to any one of the senses, is possible only in the flesh, not in a material body per se. Thought, though it does not preexist the movements, may in fact emerge from the expressions of the moving hands, arms, and body parts. The "powerlessness to stage the phenomenon . . . can be understood as our abandoning the decisive role in appearing to the phenomenon itself" (Marion, 2002, p. 132). But once produced for a first time, every repeated movement produced with an intention essentially encloses or presupposes the reminiscence of a power, or the cause (of immediate effects) that is inseparable from it (Maine de Biran, 1852/1952). It is this immemorial, unrepresented memory that represents an "I can" that underlies the intention to move, to gesture, to sense by means of the hand.

Material phenomenology and phenomenological sociology

At some time in the evolution of the human species, thought did not exist as we know it today; nor was there anything like a self-understanding of the human agent a sub-ject confronting the ob-ject, that which is thrown before it and stands over and against it. How has thought arisen in the absence of thought? Only a materialist approach can provide an answer – material life has given rise to mind rather than mind to material life. Gesticulation is produced non-consciously and necessarily has to be accompanied by speech (McNeill, 1992). Yet the hands know how to move even prior to the emergence of signifying gestures and it is unlikely that symbolizing gestures emerge without prior sensorimotor capacities (LeBaron & Streeck, 2000). In fact, one study does show how science students symbolizing gestures emerge from previous manipulative movements of their hands and arms used to bring about experiments in laboratory settings (Roth, 2003). To understand the emergence of gesticulation and the precursors of mathematical understanding, it therefore may be useful to draw on materialist approaches. Rather than grounding themselves and their constructs in reified commonsense discourses, materialist approaches are concerned with the evolutionary and cultural-historical conditions that could have led to the emergence of experiences and discourses about such phenomena as space, cubes, geometry, mind, and so on. These approaches are concerned with the phenomenologically *necessary* precursors of (mathematical) cognition. The point is that mathematical (cognitive) structures are not abstracted away from experiences but that the experiences are (immanently) present in the same way that neurons mirroring our the firing of motor neurons also are active when we see someone make the same movements.

Material phenomenology

Material phenomenology shifts the discourse from the material body to the flesh, which has been variously defined: always as an identity of two processes (e.g., seeing–seen, touching–touched, feeling–felt, hearing–heard) but most recently "as the identity . . . of the affected with the affecting" (Marion, 2002, p. 231). This flesh auto-affects itself in moving prior to any intention to move (think of the newly born child moving his arms/hands in seemingly random fashion). From this auto-affection of the flesh arises intention, a sense of "I can": an arm or hand moves because it can, the eye searches and sees because it can search and submit itself to visual impression. This is what Merleau-Ponty (1964a) has realized near the end of his life and prior to being able to take the idea further.[8] In the notes accompanying the manuscript of *Le visible et l'invisible*, he remarks having to come back to, and revising, the analysis of the cube. This analysis would show that incarnation, the flesh, is the necessary condition for a world: "My vision and my body themselves emerge from the *same* Being, which is, among others, *cube*" (p. 252, original emphasis). I do not merely perceive a cube, out there, but, in the perception, I perceive myself, my perception. It is the object that I later have come to name a cube that has taught my eyes and hands to follow its outlines so that they recognize it when seeing or feeling it again. "It is for my flesh, my body of vision, that there can be the cube itself" (p. 253). But, "the cube as such, the one with six equal faces, exists only for a non-situated gaze" (p. 252). The flesh is the precondition of a cube, because without it, there would be no experience of bodies, including the material body of the flesh, which, in touching, recognizes itself to be of the same order as the

[8] Varela, Thompson and Rosch (1991) use a long excerpt from the early Merleau-Ponty (i.e., 1945) to articulate and ground their position on the embodiment of mind, especially concerning the intertwining of intention and the object world perceived. But Merleau-Ponty revised his position after reading Maine de Biran, whose existence was pointed out to him during his doctoral defense. Henry (2000) shows how Merleau-Ponty, at the end of his life, has begun but an attempt in overcoming the opposition of the body that is feeling and the body that is felt.

other material bodies surrounding it. It is the flesh, with its capacity of tact (i.e., sense of touch), contact (i.e., touched and being touched), and contingency that is the ground of all senses, sense-making efforts, and, therefore, knowledge. The flesh, seat of the senses, also is the source of the movement undergirding sense impressions, so that movement is immanent to the activation of the senses, especially that of touch, the privilege of which is shared with all other senses (Henry, 1965). We learn to see (in) three dimensions – which we see even when presented with flat photos, drawings, pictures, and paintings – precisely because of the association of vision with the experience of movement and touch.

In the material phenomenological approach, touch is the primary sense-making instrument (Maine de Biran, 1841). It is from the movement of his fingers – which adjust themselves over the cube, envelope it at multiple points, run successively over its faces, slide with ease over its edges and follow their direction – that Chris's hands recognize the cube. There is a unique resistance with different impressions and the surface abstracts itself from the cube, "the contour from the surface, the line from the contour" (p. 31). When his hands move again over the cube, and when his finger points to the different faces that expose themselves as squares, which he names squares, they do so with the memory of previous experiences of holding the cube. The diminishing effort involved in repeating experiences "not only is the primary condition, but also the complete and abridged form of consciousness" (Ravaisson, 1838, p. 19). If Chris can recognize a cube visually, this experience itself is grounded in the coordination of the perceptual sense with that of touch. It is precisely because the hand combines motility with sensibility, precisely because it is flesh, that "it opens a quarry to intelligence and furnishes it with its most solid materials" (Maine de Biran, 1841, p. 32).

Near the beginning of my text, there is a description of Chris' "response" to the teacher's question, and I suggest that it is essentially built on (gestural) movement

and touch. In touch, there are actually two important aspects. First, there is a form of resistance that the flesh experiences to itself that has to be overcome by the effort that puts the hand in motion. Second, there is the sense of touch that affects the hand giving it the sense impressions of and from the surface, the roughness of the surfaces, the sensation of hot and cold, the way in which the surfaces curves. Traditionally, the sense impressions have been thought primarily as constituting a primordial. But in the senses, there is an intention. Yet this intention floats in other theories, because these do not indicate how such intention (to collect "information" by means of the senses) comes to be possible in the first place. It is precisely the self-affection of the flesh, the memory of the movement in the movement itself that explains the possibility of the intention. Because of the movement that experiences itself, the hand "knows" to enact the movement required to sense by means of touch. Now it is the resisting continuum that renders manifest reality. This body, the one that phenomenological philosophers now call flesh, is termed by Maine de Biran the "organic body." It is the living, organic body – i.e., the flesh – that has the powers to discover itself as a body. The flesh is first, it is *before* (intentional) sense impressions, and, therefore, it is an organic, living body that exists before the world, and before itself as body among other bodies.

Every movement that I have executed during a first time with some object, I can recognize later when I take the same solid in my hand, without that this recognition has any other condition than the movement itself, which now becomes a form of "sign," which is not really a sign. This is where the material phenomenological approach differs from others that work with bodily *schemas* that serve as the ground for abstraction and metaphorization.[9] In the approach that Maine de Biran

[9] Johnson (1987) uses the term schema in Kant's sense (see chapter 2). But for Kant, schema are in the mind, not in the body; and even if they were in the body, they could be related to the mind only because of a construction because there is no connection between body and mind in Kant's work (Nancy, 2008).

initially articulated – the one subsequently worked out by Michel Henry – there is no schema that would mediate the enactment of a movement, drive the body to move itself, it is the movement itself that serves as the sign, the power of self-movement that the effort inscribes during a first occurrence. “All movements executed by the hand, all positions that it has taken by running along the solid, may be repeated at will in the absence of the solid” (Maine de Biran, 1859b, p. 147). In fact, “the memories of an act encloses within it the sensation of the power required to repeat it” (p. 474). The next time the movement is executed, the renewed effort will be less, and the motor that has enacted the movement cannot but recognize the difference as its own will, intention. The same movement, in repeating itself, by orienting itself to the same solid, its forms and its diverse qualities, will permit me to recognize this solid without that this recognition would have any other condition than the movement itself.

This form of memory has to be distinguished from a second kind of memory, the one that we are more familiar with, a memory mediated by some sign, a form of representation, whether it is an internal one (word, image) or an external one (knot in handkerchief, note on paper). This second form of memory depends on and is enabled by the first, which is the trace of the way in which the memory was formed in the first place. Cognition is embodied precisely because of this dependence, which bridges explicit representation and intention, on the one hand, with the pre-intentional capacities of the flesh. “To say the truth, these memories themselves are nothing but the free awakening of the thought of possibilities that we say are sleeping within these, but which are of an entirely different order, foreign to the thought, to say the truth, to any representation, to any memory, which are the immemorial power of my pathic flesh” (Henry, 2000, p. 208). Chris’s memory of the cube then is immanent to movements that experience and feel themselves and move by themselves. It is here that we have to seek/find an experience that only belongs

to it, the movement. It is precisely then that we no longer have the distinction between mind and a material body – mind is the flesh itself, memory in the movement rather than in some bodily schema or representation that is used to bring the movement about. Mind does not act on the body or instructs it to do what it has to do. The possibilities in and of flesh *constitute* the most ancient form of memory.

> By laying the tangible object aside, we may conceive how the traces it has left can live again in the internal sense, by the exercise of the same activity that has concurred during their formation in the first place. And first, the module or instrument that serves to determine these forms always is present: all the movements executed by the hand, all the position that it has taken in passing over the solid, can be repeated voluntarily in the absence of the solid. These movements are *the signs of the diverse elementary perceptions* relative to the initial qualities inseparable from resistance. They can serve to recall the ideas corresponding to it, and this recall, executed by means of the available signs, constitutes the memory properly said. There is therefore a true memory of tangible forms. (Maine de Biran, 1859b, p. 147, original emphasis)

This form of thinking is important, as we require nothing but the movement itself in the absence of any representational and representative intentionality, even in the absence of the five senses. Auto-affection of the moving flesh is the precondition of the senses and, therefore, to all sense-making processes and their outcomes.

As the effort in executing the movement diminishes and the consciousness required executing it, that is, as the movement becomes a habit, this movement does not leave the sphere of intelligence (Ravaisson, 1838). “It does not become the mechanical effect of an external impulse,” such as a mental plan, “but the effect of a penchant that follows the will” (p. 28). This penchant is a tendency, which inherently implies intelligence, toward the end that the will gave to itself. In more

recent analyses – in the field of phenomenology – the ensemble of such tendencies is denoted by the term *habitus* (Bourdieu, 1980), constituting a system of structured structuring dispositions. It is a form of immediate intelligence, a sense of the game, where nothing separates subject and object. In this way, "the influence of the habitus also extends to the highest and purest regions of heart and mind" (Ravaisson, 1838, p. 40).

Phenomenological analyses of experiences with cubes and other spatial objects, from Maine de Biran to the present day, show that geometrical objects are not apprehended all at once. In this, material phenomenology is consistent with the enactivist/embodiment approach. Where they diverge is in the fundamental starting point, which for a material phenomenology lies in an originary passivity that precedes all intentionality underlying sense-making efforts of the embodied and enactive agent. Prior to the intention or will to explore the world, prior to any sense-making effort, there are experiences that allow intentionality to emerge. This fundamental passivity integral to all knowing is completely untheorized in current embodiment approaches.

Our knowledge of objects, such as the cube, is the result of a dialectical movement involving part and whole. Even the simplest figure, to be figure, requires the movement of the eye. Using a solid cube as an example, Maine de Biran (1859b) asks how an idea of the cube can precede the experience of the cube as a totality. How would it possible to have a conception a priori of all the different forms of experience that are possible with a cube? How would it be possible to have an idea of all the relationships that can be abstracted from the cube by means of analysis once we know what a cube is? To someone who has not yet learned to see, the cube would appear as squares or as whatever figure depending on the perspective. Maine de Biran then considers a blind person who learns to touch. For such a person, the idea of a cube as a whole could not exist prior to the distinct perception of its parts.

There is a true synthesis at work, as the composite whole does not precede the composition of its parts: “these two modes of generation of which we speak therefore are reduced to the same” (p. 301). When the idea of the composite has formed, it is possible to return to the simple and to consider it separately; but this simple has been the starting point for the composite whole, here, the cube. There is a memory of the simple in the series of coordinated movements, and this memory is in the flesh, which has auto-affected itself. Maine de Biran suggests that this not only is how sensory synthesis occurs but also that it is a true emblem of what occurs in all forms of reasoning properly speaking.

From an intellectualist position, thought is the seat of knowledge and intention. The question is how anything without extension, thought or consciousness, can have an effect on something as material as the body. Maine de Biran points out that it is not the body that matters, because it is the result of processes even more fundamental and ancient than the body that is said to be the seat of embodied knowledge. The original corporeity is not my body that I can touch and feel as my hand moves along it. The original corporeity is situated in the hand that does the touching and feeling. The question now is how this hand, the instrument (means) that allows us to know, has itself become known to us? This question is important, as the hand, as a body, has itself come to be known to me as a body. How is a mobile organ, the hand, able to move itself without being known? How can a hand move itself intentionally in a certain direction without an intention? It is the immanence of all its powers of my flesh in the flesh that the latter becomes the place of an immemorial memory. This is the memory of a body that remembers each time when it takes a cube, turns it in the hand, remembering not only how it feels but how it looks as a result of the turning. When I move my hand in manipulating the cube, it is not an objective organ, “it is not given to any ‘remembrance’ properly speaking, to any representation, to any thought: it is the self-movement of a prehensive power

revealed to itself in the pathic self-donation of my originary corporeity" (Henry, 2000, p. 207).

Phenomenological sociology, sociological phenomenology

Up to this point, it may have appeared as if I were moving myself into the same cul de sac where so many other scholars have ended interested in epistemological questions – how the *individual* comes to know and how this knowing relates to culture and society. This problem does not appear to be solved by the current literature on embodiment, which tends to substantialize knowing in the individual body (Fr. *corps*) independent of its relation to the collective body of society (Fr. *corps social*).[10] There is in fact a close relation between the social and material space that comprehend the individual and the comprehension of this individual of the social and material space (Bourdieu, 1997). Any bodily experience, therefore, is social experience; and any social experience is bodily experience.

The comprehension of the social and material space initially and above all is practical (i.e., not constructed), occurs in collectively motivated practical activity, emerging from engagement with the (material, social) beings that populate this space. It is because of the "material inclusion – often unnoticed and suppressed – and what follows from it, that is, the incorporation of social structures in the form of dispositional structures . . . that I acquire a practical knowledge and mastery of the encompassing space" (p. 157). It is therefore unlikely that Chris would come to rediscover geometry entirely on his own, which, by him, would have to be "taken-as-

[10] The position taken here differs from that in the enactivist literature. Varela et al. (1991) propose to look for knowledge at the "interface between body, society, and culture" (p. 179). In the position articulated here there is no interface: mind *is* in society and culture as much as society and culture are in the mind. Similar positions can be found in activity theory from L. S. Vygotsky via A. N. Leont'ev to the present day. Maturana and Varela (1980) take societies to be "systems of coupled human beings" (p. 118), whereas the position here is the converse: the specifically *human* being is a result of society rather than preceding coupling or, in activity theoretic terms, there is mind because there is society.

shared" with others if we were looking at the situation a social constructivist perspective. But Chris has come to know the cube as cube as the result of the multiple relations in which his interactions with others made specific structures salient. This salience is a result of interactions, of which the distinction between Chris, as personality, and the world is another one of the results (Leontjew, 1982). That is, material space becomes salient within social space, which is both a condition and a result of *collective* activities in which each human being participates. The body is the principle of collectivization:

> Having the (biological) property of being open to the world, thus exposed to the world, and, thereby, susceptible to be conditioned by the world, shaped by the material and cultural conditions of existence in which it is placed from the beginning, it is subject to a process of socialization of which individuation is itself the product, the singularity of the "me" being fashioned in and by social relations. (Bourdieu, 1997, p. 161)

That this is not recognized generally is a problem that arises with the category of the *body*, which is a thing among things, leading to the perception that the human (social) sciences can be transformed into natural sciences. If there is a difference between the living and the material body of a person, then there must be *distinct* categories, lest there be conceptual confusion. Phenomenological philosophers introduced the category of the flesh precisely to make this vital distinction. Thus, there is a tight integration between the two forms of spaces, as "social space tends to be translated into physical space" (p. 161), and, conversely, physical space is translated into social space (e.g., physical space in church, courts of law, school, and so on are the result of and resources in the exercise of power). Consequently,

> [t]he world is comprehensible, immediately endowed with sense, because the body – which, because of its senses and its brain, has the capacity to be present at the outside of itself, in the world, and to be impressed and durably

> modified by it – has been for a long time (since the beginning) exposed to its regularities. (p. 163)

As soon as we read Bourdieu's body (*corps*) as flesh, because of the senses of the body enabled by auto-affection, his theory of the body being a socialized social body can be integrated with the materialist phenomenological analyses of the flesh and incarnation.

From this exposure to regularities – any social regularity, such as speech, itself being associated with a material regularity (sound) – that are engaged by means of corporeal knowledge arises a practical comprehension that is very different from "the intentional act of conscious decoding that is normally designated by the idea of comprehension" (p. 163). To understand the practical comprehension of the world, one has to abandon mentalism and intellectualism – without abandoning the possibility that there can be non-conceptual forms of organization in which language does not intervene (e.g., ability to form grammatically correct sentences without knowledge of formal grammar). To comprehend practical comprehension, Bourdieu suggests, we need to pursue alternatives to all forms of ("idealist" and embodied) constructivism. These alternatives have to allow for generative agency, but this agency is that of a "socialized body, which invests in its practice socially constructed organizational principles that have been acquired in the course of a socially situated and dated experience" (p. 164).

We come to have social bodies because we participate from our beginnings in collective activities that take place in a world shot through with significations. It is in participating that we develop a practical sense of how the (social, material) world works. This "practical sense is that which allows to act as required . . . without posing or executing a 'you must,' a rule of conduct" (p. 166). It constitutes a practical sense inhabited by the world it inhabits. It is precisely for this reason that Hans Freudenthal (1983) could write, "[w]ith much eloquence I have exerted myself to

convince the reader of how the mental objects of Euclidean geometry are forced upon us, from bow and arrow to baby spoon to television aerial" (p. 228). Habitus allows us to recognize a situation as making sense based on a practical, corporeal sense of anticipation of the tendencies of the field in which such things as boxes (right parallelepipeds) appear. The practical knowledge arises not from an external relation to the world.

"I chose the term 'boxes' for the geometric object I am going to consider (right parallelepipeds)," writes Freudenthal (1983), "because this article is available in the richest variety" (p. 229), and they are equally "forced upon us" from the beginning as the other objects. The pizza box and cube in Chris' hands are not just material things out there; they are integral aspects of the patterns in a *specifically human* society. At first, there is a practical comprehension, and this practical comprehension we can see in the episode. Chris moves his left hand along one of the sides of the pizza box, then along its other side, neither one of which is square. He subsequently takes the cube, points to its different faces, saying that all of them are square. Even if he does not yet verbally articulate a response in some abstract way, for example, by saying "to get a cube, you need to stack a few pizza boxes," we see in the series of actions that point to the non-square sides of one object followed by the comment that the cube has all sides square. The practical sense of the habitus – his habitus being but a concrete realization of collective habitus, formed by the collective field – is the principle of practical comprehension; and this habitus is inhabited by the world that inhabits it. It guarantees an immediate relation of engagement, by means of the senses, which gives sense to the world. In the same way that intentionality emerges from the "I can" of the motor system of the flesh, sense, collective intentionality, emerges from the way in which habitus can anticipate the changes in the field of which it is an integral part. All of this is

common sense precisely because we have senses in common, because we are flesh before we are bodies.

Toward a conceptualization of mathematics in the flesh

Embodiment has become one of the fashionable theories in mathematics education. Yet many mathematics educators still have problems with this approach, which may arise from their sense that this literature insufficiently articulates the mechanisms by means of which the body – or, rather, the flesh – is *always* involved in "cognition" not merely a stepping-stone to abstract intellectualism. Throughout this text, I emphasize a categorical distinction that presently is not made in the mathematics education literature: that between material bodies and the human body, a distinction that phenomenological philosophers make between the *body* (Ger. *Körper*, Fr. *corps*) and something sometimes translated as the *flesh* (Ger. *Leib*, Fr. *chair*). The contribution to specifically human cognition is made by the flesh (*Leib, chair*) rather than by the body (*Körper, corps*), which is the aspect of the flesh indistinguishable from other bodies. But knowing is due to the flesh, not due to the body: there are many bodies that populate this world that do not have the special feature that we refer to as knowledge. The body (*corps*), the presence of which is enabled by the flesh, is inherently a socialized body (*corps socialisé*) because it is shaped in and by society (*corps social*).

In this paper, I focus only on the experience, the relation of the objects and the nonverbal phenomena of touch and gesture. The question remains, however, how something like a cube comes to be integrally related to the verbal articulation, verbal thought, and the word. A way to the answer of this question has already been sketched. Writing and speaking are the two incarnated modes of language that are closely related and in fact interlaced with the hand, which constitutes the essential

connection between humans and their world, and therefore also embodies intentionality (Heidegger, 1982). Thus, the gestures of the hand could never come to be interwoven with language unless the word and the hand already had something in common (Franck, 1986). “There is no voice except a carnal voice, and our whole body is what it presupposes to be what it is, and what, in return, the voice makes into its mouthpiece” (Chrétien, 1992, p. 101). That which they do have in common is that they are expressions of an originary, pre-linguistic, pre-intentional, and pre-ontological “I can.”

Mathematics educators often pay lip service to grounding mathematics teaching and learning in what and how students actually know and experience. Despite all the rhetoric about the importance of students’ experiences in the learning of mathematics, most mathematics educators do not provide evidence that they are actually concerned with understanding the world through the eyes of the learner. But if we want to understand learning, especially, what students can rationally intend given that they do not know the object of their learning, we need a radical, material phenomenology, one that begins by investigating how the world looks like to students and what kinds of world they currently inhabit. We need a way to describe what learners are conscious off and what is present to their experience – and mental constructions are not of this kind. From the movements that Chris deploys, including the ones that make the air move so we can hear “cube,” we *cannot* infer that he has some “mathematical understanding” in the traditional sense. But in his flesh, auto-affection leaves an immemorial memory that underlies all sensing and sense making: It is because of the flesh that the sense of the body, the bodily sense, becomes the body of sense.

References

Alibali, M. W. and Goldin-Meadow, S. (1993) 'Gesture-speech mismatch and mechanisms of learning: What the hands reveal about a child's state of mind', *Cognitive Psychology* **25**, 468–523.

Bourdieu, P. (1980) *Le sens pratique*, Paris: Les Éditions de Minuit.

Bourdieu, P. (1997) *Méditations pascaliennes*, Paris: Seuil.

Chrétien, J-L. (1992) *L'appel et la réponse*, Paris: Les Éditions de Minuit.

Cobb, P. (1999) 'Individual and collective mathematical development: The case of statistical data analysis', *Mathematical Thinking and Learning* **1**, 5–43.

Derrida, J. (2005) *On touching – Jean-Luc Nancy*, Stanford, CA: Stanford University Press.

Franck, D. (1986) *Heidegger et le problème de l'espace*, Paris: Les Éditions de Minuit.

Freudenthal, H. (1983) *Didactical phenomenology of mathematical structures*, Dordrecht, The Netherlands: D. Reidel.

Harnad, S. (1990) 'The symbol grounding problem', *Physica D* **42**, 335–346.

Heidegger, M. (1982) *Parmenides*, Frankfurt: Vittorio Klostermann.

Henry, M. (1965) *Philosophie et phénoménologie du corps: Essai sur l'ontologie biranienne*, Paris: Presses Universitaires de France.

Henry, M. (2000) *Incarnation: Une philosophie de la chair*, Paris: Éditions du Seuil.

Johnson, M. (1987) *The body in the mind: The bodily basis of meaning, imagination, and reason*, Chicago: University of Chicago Press.

Lakoff, G. and Núñez, R. (2000) *Where mathematics comes from: How the embodied mind brings mathematics into being,* New York: Basic Books.

LeBaron, C. D. and Streeck, J. (2000) 'Gestures, knowledge, and the world', in D. McNeill (ed.), *Language and gesture,* Cambridge: Cambridge University Press, pp. 118–140.

Leontjew, A. N. (1982) *Tätigkeit, Bewusstsein, Persönlichkeit,* Köln: Pahl-Rugenstein.

Maine de Biran, P. (1841) *Œuvres philosophiques, tome premier: Influence de l'habitude sur la faculté de penser,* Paris: Librairie de Ladrange.

Maine de Biran, P. (1859a) *Ouvres inédites, tome I*, Paris: Dezobry & Magdeleine.

Maine de Biran, P. (1859b). *Ouvres inédites, tome II*, Paris: Dezobry & Magdeleine.

Maine de Biran, P. (1852/1952) *Mémoire sur la décomposition de la pensée,* Paris: Presses Universitaires de France.

Marion, J-L. (2002) *Being given: Toward a phenomenology of givenness*, Stanford, CA: Stanford University Press.

Maturana, H. R. and Varela, F. J. (1980) *AUtopoiesis and cognition: The realization of the living,* Dordrecht, Holland: D. Reidel.

McNeill, D. (1992) *Hand and mind: What gestures reveal about thought*, Chicago: University of Chicago.

McNeill, D. (2002) 'Gesture and language dialectic', *Acta Linguistica Hafniensia* **34**, 7–37.

Merleau-Ponty, M. (1945) *Phénoménologie de la perception*, Paris: Gallimard.

Merleau-Ponty, M. (1964a) *Le visible et l'invisible*, Paris: Gallimard.

Merleau-Ponty, M. (1964b) *L'œil et l'esprit*, Paris: Gallimard.

Nancy, J-L. (2006) *Corpus*, Paris: Métailé.

Nancy, J-L. (2008) *The discourse of the syncope: Logodaedalus*, Stanford, CA: Stanford University Press.

Pinxten, R. (1991) 'Some remarks on Navajo geometry and piagetian genetic theory', *Infancia y Aprendizaje* **54**, 41–52.

Ravaisson, F. (1838) *De l'habitude*, Paris: H. Fournier.

Roth, W.-M. (2003) 'From epistemic (ergotic) actions to scientific discourse: Do gestures obtain a bridging function?', *Pragmatics & Cognition* **11**, 139–168.

Roth, W.-M. and Thom, J. (2009) 'The emergence of 3d geometry from children's (teacher-guided) classification tasks', *Journal of the Learning Sciences* **18**, 45–99.

Varela, F. J., Thompson, E. and Rosch, E. (1991) *The embodied mind: Cognitive science and human experience*, Cambridge, MA: MIT Press.

von Glasersfeld, E. (1989) 'Cognition, construction of knowledge, and teaching', *Synthese* **80**, 121–140.

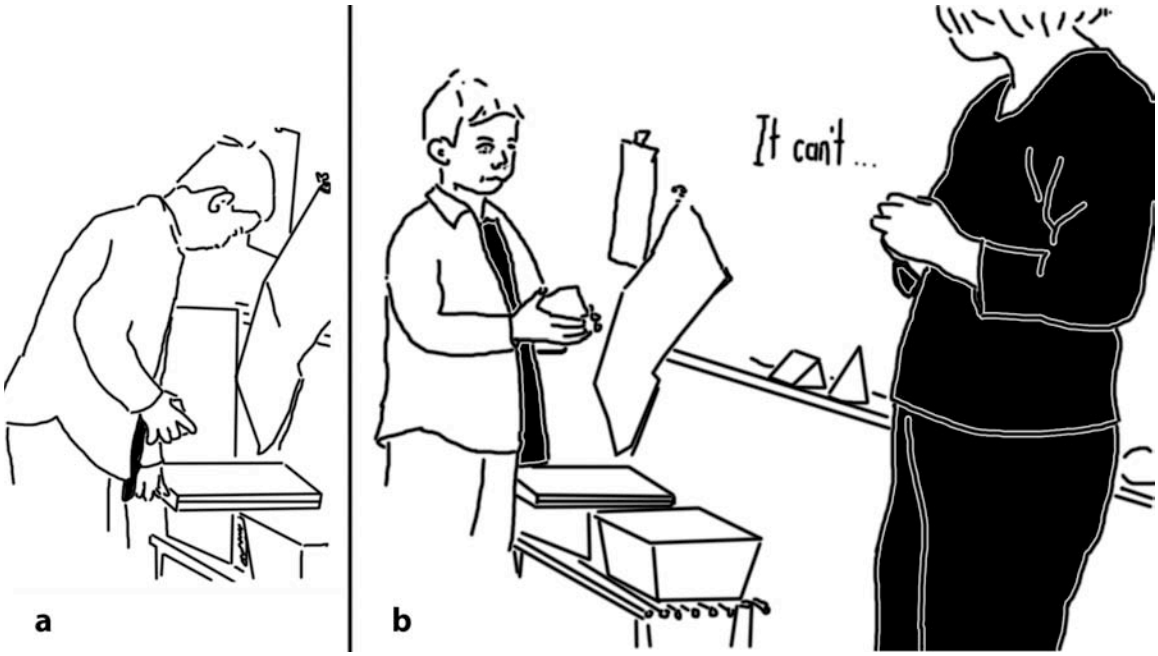


Figure 1. a. Chris is oriented to the box following what we can hear to be a teacher question, “What would that box have to have to be a cube?” b. At the end of the presentation, Chris squarely gazes at his teacher, no longer “caught up” with the pizza box but into the familiar terrain of the red cube that he has manipulated so many times before.