3. GENERAL EQUILIBRIUM
AND THE WELFARE THEOREMS

OUTLINE

3.1 THE EXCHANGE ECONOMY REVISITED

Recall the simple two-person exchange economy from Topic 2, depicted in Figure 3-1.

In our discussion of that economy in Topic 2 we made no mention of trade (despite calling it an "exchange" economy); our focus was on the characterization of Pareto efficiency.

We now want to examine trade in this economy.

Suppose person 1 currently has amounts $\overline{X}_1$ and $\overline{Y}_1$. This is his **endowment**. Similarly, person 2 has endowment $\overline{X}_2$ and $\overline{Y}_2$. See Figure 3-2.

Now suppose each person can buy or sell *X* and *Y* at prices p_X and p_Y respectively. This means that their current holdings of *X* and *Y* now have a *market value*.

In particular, the market value of person 1's endowment is

$$M_1 = p_X \overline{X}_1 + p_Y \overline{Y}_1$$

We call this his **wealth**.

Similarly, the wealth of person 2 is the market value of her endowment:

$$M_2 = p_X \bar{X}_2 + p_Y \bar{Y}_2$$

Because trade is now possible, each person has the freedom to choose a consumption bundle different from their endowment bundle by buying and selling goods such that the market value of their consumption bundle is just equal to their wealth.

That is, each person can choose their consumption bundle to maximize their utility subject to a wealth constraint.

<u>The choice problem for person 1</u>

$$\max_{x_1, y_1} u_1(x_1, y_1) \text{ subject to } p_X x_1 + p_Y y_1 = M_1$$

Graphically, the wealth constraint for person 1 is a line with slope

$$-\frac{p_X}{p_Y}$$

passing through his endowment point. See Figure 3-3.

Why? If he sells his entire endowment of *Y* and uses the proceeds to buy more *X*, he can buy an amount

$$\frac{p_Y \bar{Y}_1}{p_X}$$

and thereby consume the maximum possible amount of *X* that he can afford:

$$X_1^{MAX} = \bar{X}_1 + \frac{p_Y \bar{Y}_1}{p_X}$$

This is the horizontal intercept of the wealth constraint in Figure 3-3.

Conversely, if he sells his entire endowment of *X* and uses the proceeds to buy more *Y*, he can buy an amount

$$\frac{p_X \bar{X}_1}{p_Y}$$

and thereby consume the maximum possible amount of *Y* he can afford

$$Y_1^{MAX} = \bar{Y}_1 + \frac{p_X \bar{X}_1}{p_Y}$$

This is the vertical intercept of the wealth constraint in Figure 3-3.

The slope of the constraint is the just the ratio of the vertical intercept and the horizontal intercept (but its negative), and that reduces to

$$-\frac{Y_1^{MAX}}{X_1^{MAX}} = -\frac{p_X}{p_Y}$$

The solution to the choice problem for person 1 is the usual tangency condition:

$$MRS^1 = \frac{p_X}{p_Y}$$

See Figure 3-4. In the case illustrated, person 1 is a seller of *X* and a buyer of *Y*.

Note that the market value of what he buys must be exactly equal to the market value of what he sells (or else he would not be on his wealth constraint). That is,

$$p_Y (Y_1^* - \bar{Y}_1) = p_X (\bar{X}_1 - X_1^*)$$

<u>The choice problem for person 2</u>

$$\max_{x_2, y_2} u_2(x_2, y_2) \quad \text{subject to} \quad p_X x_2 + p_Y y_2 = M_2$$

The graphical representation of this problem is illustrated in Figure 3-5. The wealth constraint for person 2 is a line with slope

$$-\frac{p_X}{p_Y}$$

passing through her endowment point, and the solution to the choice problem is a tangency condition:

$$MRS^2 = \frac{p_X}{p_Y}$$

In the case illustrated, person 2 is a seller of *Y* and a buyer of *X*.

Note that the market value of what she buys must be exactly equal to the market value of what she sells (or else she would not be on her wealth constraint). That is,

$$p_X\,(X_2^* - \overline{X}_2) = p_Y\,(\overline{Y}_2 - Y_2^*)$$

<u>Equilibrium</u>

Can the consumption points illustrated in Figures 3-4 and 3-5 jointly constitute an equilibrium?

Clearly not, because the amount of *X* that person 2 wants to buy is much more than the amount that person 1 wants to sell, and the amount of *Y* that person 2 wants to sell is much more than the amount that person 1 wants to buy.

That is, supply and demand are not equated at the candidate prices; there is an excess aggregate demand for *X* and an excess aggregate supply of *Y*.

Intuitively, this imbalance of supply and demand can only be corrected by an increase in the price of *X* relative to the price of *Y*, thereby dampening demand for *X* relative to *Y*.

That correction of relative prices must continue until equilibrium is achieved, where

$$\bar{X}_1 - X_1^* = X_2^* - \bar{X}_2 \quad \text{and} \quad Y_1^* - \bar{Y}_1 = \bar{Y}_2 - Y_2^*$$

or equivalently, where

$$\bar{X}_1 + \bar{X}_2 = X_1^* + X_2^* \quad \text{and} \quad \bar{Y}_1 + \bar{Y}_2 = Y_1^* + Y_2^*$$

That is, the total quantity available must be equal to the amount consumed, for both goods *X* and *Y*. This equilibrium is illustrated in Figure 3-6.

<u>Welfare Properties of the Equilibrium</u>

The individual decision-making of the two people in this economy lead to an outcome where

$$MRS^1 = \frac{p_X}{p_Y} \quad \text{and} \quad MRS^2 = \frac{p_X}{p_Y}$$

Since they face the <u>same price ratio</u>, these two conditions imply that

$$MRS^1 = MRS^2$$

Recall from Topic 2 that this is the condition for Pareto efficiency in this economy. Thus, the equilibrium is Pareto efficient.

This property of the exchange-economy equilibrium reflects a far more general result that we will encounter in Section 3.4: the first welfare theorem.

We can actually make an even stronger statement about the equilibrium here: it is **in the core with respect to the endowment point**. How do we know this?

We know that neither person has to trade if they choose not too; they could simply consume their endowment. Thus, these people will trade if and only if doing so leaves neither of them worse off than at their endowment, and at least one of them better off (or else why bother trading at all). That is, if trade does occur than it must yield a Pareto improvement over the endowment point; the equilibrium Pareto-dominates the endowment point.

Since the equilibrium is Pareto efficient (because the MRSs are equated) and Pareto-dominates the endowment point, then by definition of the core, the equilibrium must be in the core with respect to the endowment point; see Figure 3-7.

3.2 EXAMPLE: SOLVING FOR THE COMPETITIVE EQUILIBRIUM OF AN EXCHANGE ECONOMY

Let us begin by summarizing the equilibrium conditions we derived in Section 3.1:

(3.1) $$MRS^1 = \frac{p_X}{p_Y}$$

(3.2) $$MRS^2 = \frac{p_X}{p_Y}$$

(3.3) $$p_X X_1^* + p_Y Y_1^* = p_X \bar{X}_1 + p_Y \bar{Y}_1 \equiv M_1$$

(3.4) $$p_X X_2^* + p_Y Y_2^* = p_X \bar{X}_2 + p_Y \bar{Y}_2 \equiv M_2$$

(3.5) $$Y_1^* - \bar{Y}_1 = \bar{Y}_2 - Y_2^*$$

(3.6) $$\bar{X}_1 - X_1^* = X_2^* - \bar{X}_2$$

The first two equations are the tangency conditions. The next two equations are the wealth constraints. The final two equations are the demand = supply equations.

Our goal is to find the simultaneous solution to these equations.

It is simpler than it might appear. To see why, first note that among equations (3.3) – (3.6) there are only three *independent* equations.

In particular, rearrange (3.3) to obtain

(3.7) $$Y_1^* - \bar{Y}_1 = \frac{p_X}{p_Y}(\bar{X}_1 - X_1^*)$$

and rearrange (3.4) to obtain

(3.8) $$\bar{Y}_2 - Y_2^* = \frac{p_X}{p_Y}(X_2^* - \bar{X}_2)$$

If equations (3.7) and (3.8) hold, and equation (3.5) also holds, then equation (3.6) must automatically hold as well. Thus, we can drop equation (3.6) from our list without losing any information.

A second simplifying transformation is also evident from our rearrangement of equations (3.3) and (3.4) into (3.7) and (3.8): prices now appear in our list of equations only in *ratio* form.

This means that we will only be able to solve for the price ratio; not the two prices independently. In general, equilibrium identifies *relative prices* but not absolute prices. (Indeed, *all* values in economics are relative; there are no absolute values).

Since we can only find the equilibrium price ratio, we can simplify our equations further by fixing $p_Y = 1$ and focus on solving for p_X. (This makes *Y* the **numeraire**).

We are now left with the following five key independent equations:

(3.9) $$MRS^1 = p_X$$

(3.10) $$MRS^2 = p_X$$

(3.11) $$Y_1^* - \bar{Y}_1 = p_X(\bar{X}_1 - X_1^*)$$

(3.12) $$\bar{Y}_2 - Y_2^* = p_X(X_2^* - \bar{X}_2)$$

(3.13) $$Y_1^* - \bar{Y}_1 = \bar{Y}_2 - Y_2^*$$

Now suppose that preferences are Cobb-Douglas for both persons:

(3.14) $$u_1(x_1, y_1) = x_1^{a_1} y_1^{b_1}$$

(3.15) $$u_2(x_2, y_2) = x_2^{a_2} y_2^{b_2}$$

Recall from Topic 2.5 that the MRS for Cobb-Douglas preferences is very simple. In particular,

(3.16) $$MRS^1 = \frac{a_1 y_1}{b_1 x_1}$$

for person 1, and

(3.17) $$MRS^2 = \frac{a_2 y_2}{b_2 x_2}$$

for person 2.

Thus, equations (3.9) and (3.10) become

(3.18) $$\frac{a_1 Y_1^*}{b_1 X_1^*} = p_X$$

and

(3.19) $$\frac{a_2 Y_2^*}{b_2 X_2^*} = p_X$$

These can in turn be rearranged as

(3.20) $$Y_1^* = \frac{p_X b_1 X_1^*}{a_1}$$

and

(3.21) $$Y_2^* = \frac{p_X b_2 X_2^*}{a_2}$$

Now substitute (3.20) into (3.11) and solve for X_1^* to obtain

(3.22) $$X_1^* = \frac{a_1(\overline{Y}_1 + p_X \overline{X}_1)}{p_X(a_1 + b_1)}$$

and then substitute (3.22) back into (3.20) to obtain

(3.23) $$Y_1^* = \frac{b_1(\overline{Y}_1 + p_X \overline{X}_1)}{a_1 + b_1}$$

We have found the consumption bundle for person 1 as a function of p_X.

Similarly, substitute (3.21) into (3.12) and solve for X_2^* to obtain

(3.24) $$X_2^* = \frac{a_2(\overline{Y}_2 + p_X \overline{X}_2)}{p_X(a_2 + b_2)}$$

and then substitute (3.24) back into (3.21) to obtain

(3.25) $$Y_2^* = \frac{b_2(\overline{Y}_2 + p_X \overline{X}_2)}{a_2 + b_2}$$

We have found the consumption bundle for person 2 as a function of p_X.

It is worth noting that the term in brackets in the numerator of expressions (3.22) – (3.25) is wealth (because remember that we have set $p_Y = 1$). Thus, for both persons, consumption of *X* and *Y* is increasing in wealth; *X* and *Y* are *normal goods* for these people.

Equation (3.22) – (3.25) describe the consumption bundles for these two people as a function of p_X. Our next step is to find the value of p_X that ensures these consumption values add up to the available goods in this economy; that is, demand equals supply.

To find this equilibrium price, we substitute (3.23) and (3.25) into (3.13) and solve for p_X:

(3.26) $$p_X^* = \frac{a_1(a_2 + b_2)\overline{Y}_1 + a_2(a_1 + b_1)\overline{Y}_2}{b_1(a_2 + b_2)\overline{X}_1 + b_2(a_1 + b_1)\overline{X}_2}$$

This tells us that the equilibrium price is a function of the preference parameters of both people, and the endowments of those people.

In the very special case where the two persons have identical preferences and where they weight both goods equally (that is, where $a_1 = a_2 = b_1 = b_2$) , expression (3.26) reduces to

(3.27) $$p_X^* = \frac{\bar{Y}_1 + \bar{Y}_2}{\bar{X}_1 + \bar{X}_2}$$

That is, the relative price of *X* simply reflects its scarcity relative to *Y*, and the *distribution* of resources at the endowment has no influence on that price.

In all other cases, the preference parameters and the distribution of resources play a role.

We can now complete our characterization of the equilibrium by evaluating the consumption bundles from (3.22) to (3.25) at the equilibrium price p_X^* . (This will give us point C in Figure 3.7).

Substituting (3.26) into (3.22) yields

(3.28) $$X_1^{**} = \frac{a_1((a_2 + b_2)\bar{X}_1\bar{Y}_1 + a_2\bar{X}_1\bar{Y}_2 + b_2\bar{Y}_1\bar{X}_2)}{a_1(a_2 + b_2)\bar{Y}_1 + a_2(a_1 + b_1)\bar{Y}_2}$$

and substituting (3.26) into (3.23) yields

(3.29) $$Y_1^{**} = \frac{b_1((a_2 + b_2)\bar{X}_1\bar{Y}_1 + a_2\bar{X}_1\bar{Y}_2 + b_2\bar{Y}_1\bar{X}_2)}{b_1(a_2 + b_2)\bar{X}_1 + b_2(a_1 + b_1)\bar{X}_2}$$

where the "**" subscript here indicates equilibrium consumption.

We can then obtain X_2^{**} and Y_2^{**} simply by noting that person 2 must consume whatever person 1 does not consume out of the available goods. That is,

(3.30) $$X_2^{**} = (\bar{X}_1 + \bar{X}_2) - X_1^{**}$$

and

(3.31) $$Y_2^{**} = (\bar{Y}_1 + \bar{Y}_2) - Y_1^{**}$$

The critical point to take away from these expressions for equilibrium consumption is that even in this very simple economy, there are highly complicated inter-relationships between the behavior of the constituent agents. In particular, the equilibrium consumption bundle for person 1 depends on her preference parameters and her endowment, and on the preference parameters and the endowment of the other person, in a highly non-linear way. Even simple economies are very complicated systems.

Walras' Law

Recall that our original six equations in (3.1) – (3.6) could be reduced to five independent equations, from which we could then find the equilibrium price ratio. The fact that there are only five independent equations in that original set of six is a manifestation of **Walras' Law**, which tells us that if we have *n* markets, and $n-1$ of those markets are in equilibrium, then the remaining market must be in equilibrium too. We will discuss Walras' Law more fully in Topic 3.3.

Welfare Properties of the Equilibrium

We know from (2.20) in Topic 2 that the Pareto frontier for an exchange economy with Cobb-Douglas preferences is given by

$$y_1^{PF} = \frac{a_2 b_1 \overline{Y} x_1}{(a_2 b_1 - a_1 b_2) x_1 + a_1 b_2 \overline{X}} \tag{3.32}$$

where $\overline{Y} = \overline{Y}_1 + \overline{Y}_2$ is the aggregate amount of good *Y*, and $\overline{X} = \overline{X}_1 + \overline{X}_2$ is the aggregate amount of good *X*.

If our reasoning from Section 3.1 is correct – and it is – then the competitive equilibrium should lie on this frontier. To check that it does, we simply substitute X_1^{**} from (3.28) for x_1 in (3.32) and confirm that

$$y_1^{PF} \Big|_{x_1 = X_1^{**}} = Y_1^{**} \tag{3.33}$$

The algebra is a bit messy but it is straightforward to make this confirmation.

A Numerical Example

Consider a two-person exchange economy with two goods in fixed amounts $X = 75$ and $Y = 150$. Person 1 has preferences represented by

(3.34) $$u_1 = x_1 y_1^2$$

and person 2 has preferences represented by

(3.35) $$u_2 = x_2 y_2$$

The endowment is E = { $\bar{X}_1 = 25$, $\bar{Y}_1 = 50$, $\bar{X}_2 = 50$, $\bar{Y}_2 = 100$ }.

First determine the MRSs for these agents:

(3.36) $$MRS^1 = \frac{a_1 y_1}{b_1 x_1} = \frac{y_1}{2x_1}$$

(3.37) $$MRS^2 = \frac{a_2 y_2}{b_2 x_2} = \frac{y_2}{x_2}$$

Now recall the five key equations (3.9) – (3.13) that define the equilibrium, which we repeat here as

(3.38) $$MRS^1 = p_X$$

(3.39) $$MRS^2 = p_X$$

(3.40) $$Y_1^* - \bar{Y}_1 = p_X (\bar{X}_1 - X_1^*)$$

(3.41) $$\bar{Y}_2 - Y_2^* = p_X (X_2^* - \bar{X}_2)$$

(3.42) $$Y_1^* - \bar{Y}_1 = \bar{Y}_2 - Y_2^*$$

In the context of our numerical example, these five equations become

(3.43) $$\frac{Y_1^*}{2X_1^*} = p_X$$

(3.44) $$\frac{Y_2^*}{X_2^*} = p_X$$

(3.45) $$Y_1^* - 50 = p_X (25 - X_1^*)$$

(3.46) $$100 - Y_2^* = p_X (X_2^* - 50)$$

(3.47) $$Y_1^* - 50 = 100 - Y_2^*$$

Make Y_1^* the subject of (3.43) and substitute this into (3.45). Solve for X_1^* to obtain

(3.48) $$X_1^* = \frac{50 + 25p_X}{3p_X}$$

Now substitute (3.48) back into (3.43) to obtain

(3.49) $$Y_1^* = \frac{100 + 50p_X}{3}$$

Similarly, make Y_2^* the subject of (3.44) and substitute this into (3.46). Solve for X_2^* to obtain

(3.50) $$X_2^* = \frac{50 + 25p_X}{p_X}$$

Now substitute (3.50) back into (3.44) to obtain

(3.51) $$Y_2^* = 50 + 25p_X$$

We have now found the consumption bundles for each person as functions of p_X. Our next step is to find the value of p_X that ensures these consumption values add up to the available goods in this economy; that is, demand equals supply.

Substitute (3.49) and (3.51) into (3.47) and solve for p_X:

(3.52) $$p_X^* = \frac{8}{5}$$

This is the equilibrium price ratio in this economy.

We now find the equilibrium consumption bundles by substituting $p_X^* = \frac{8}{5}$ into equations (3.48) – (3.51) to yield:

(3.53) $$X_1^{**} = \frac{75}{4} < \overline{X}_1$$

(3.54) $$Y_1^{**} = 60 > \overline{Y}_1$$

(3.55) $$X_2^{**} = \frac{225}{4} > \overline{X}_2$$

(3.56) $$Y_1^{**} = 90 < \overline{Y}_2$$

Thus, person 1 is a seller of *X* and a buyer of *Y*, and person 2 is a seller of *Y* and a buyer of *X*. See Figure 3-8 (which illustrates only the lower SW corner of the Edgeworth box so as to keep the diagram simple enough to read).

<u>Welfare Properties</u>

Note that Figure 3-8 depicts the equilibrium consumption point as lying on the Pareto frontier. Now let us confirm that this is indeed true.

Setting $MRS^1 = MRS^2$ from (3.36) and (3.37), and imposing the resource constraints yields

(3.57) $$\frac{y_1}{2x_1} = \frac{150 - y_1}{75 - x_1}$$

Solving for y_1 yields the Pareto frontier:

(3.58) $$y_1^{PF} = \frac{300x_1}{x_1 + 75}$$

Setting $x_1 = X_1^{**} = \frac{75}{4}$ in (3.58) yields

(3.59) $$y_1^{PF}\Big|_{x_1 = X_1^{**}} = 60 = Y_1^{**}$$

Thus, the equilibrium lies on the Pareto frontier, as depicted in Figure 3-8.

Finally, we can confirm that the competitive equilibrium lies in the core. Utility for person 1 at the endowment is

(3.60) $$u_1^E = (25)(50)^2 = 62500$$

In comparison, utility for person 1 at the competitive equilibrium is

(3.61) $$u_1^{**} = (\frac{75}{4})(60)^2 = 67500$$

Thus, person 1 is strictly better off at the competitive equilibrium.

Similarly, utility for person 2 at the endowment is

(3.62) $$u_2^E = (50)(100) = 5000$$

In comparison, utility for person 2 at the competitive equilibrium is

(3.63) $$u_2^{**} = (\frac{225}{4})(90)^2 = 5062.5$$

Thus, person 2 is strictly better off at the competitive equilibrium.

It follows that the competitive equilibrium is a Pareto improvement over the endowment. Since the competitive equilibrium is also on the Pareto frontier, it follows that the competitive equilibrium is in the core respect to the endowment.

Gains from Trade

It is perhaps tempting to compare the utility gain for persons 1 and 2 in this economy and argue that person 1 has gained more. This argument is of course false. We cannot compare utility across individuals; the cardinal value of each utility has no meaning.

To calculate the gains from trade – and how they are split between the two persons – we would need to calculate the **compensating variations** for the two agents but that is beyond our coverage in this course.

End of Part 1

See Part 2 for Sections 3.3 and 3.4

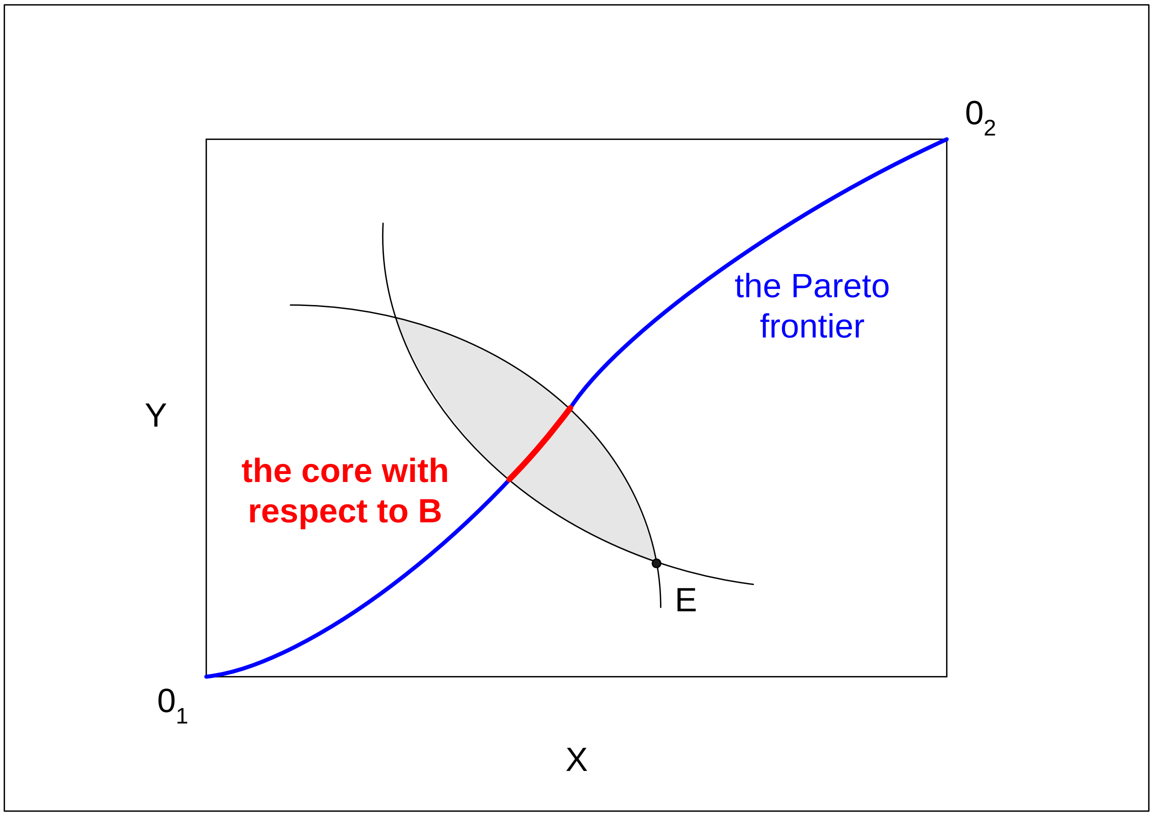

Figure 3-1

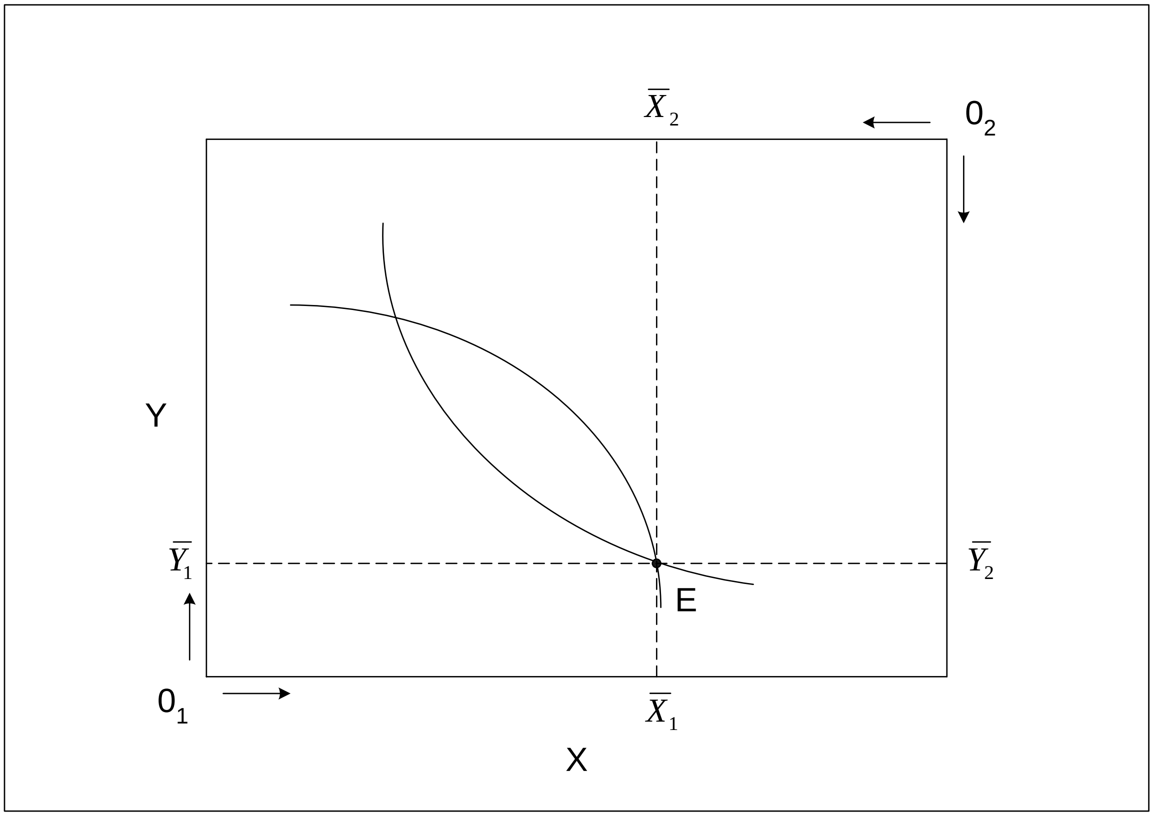

Figure 3-2

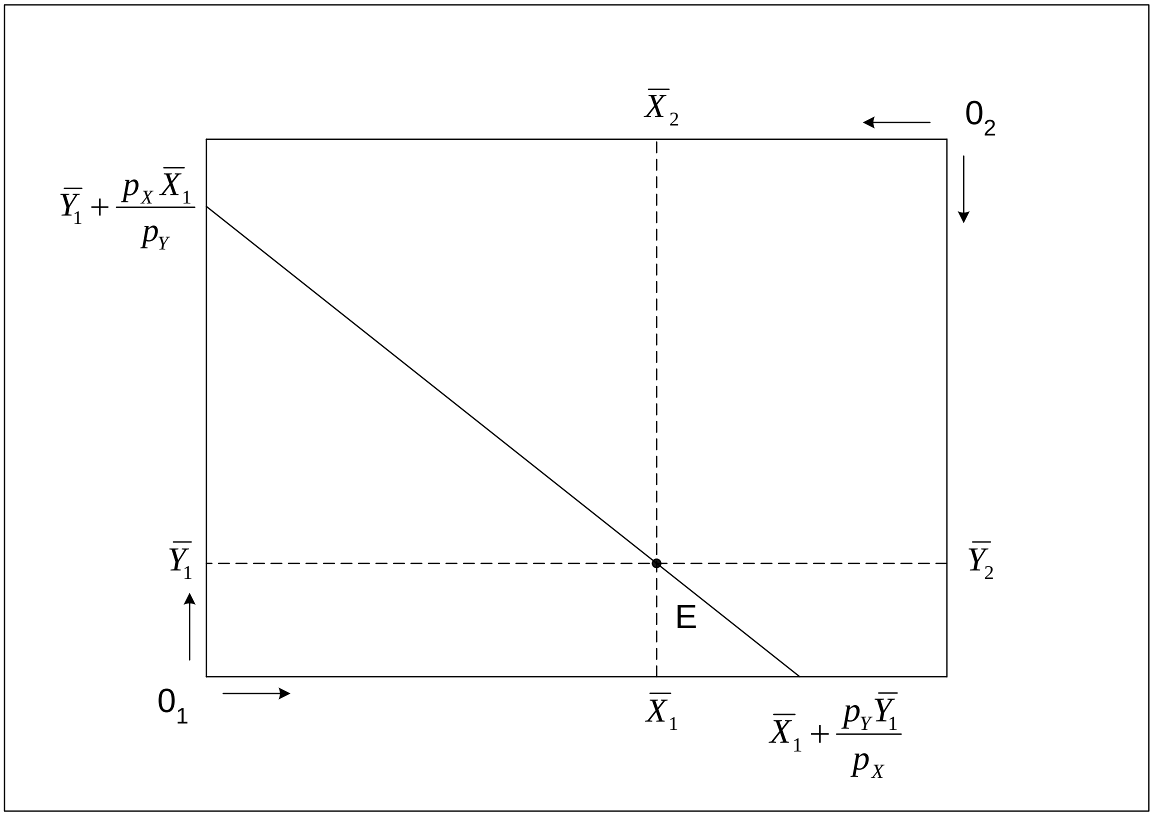
$\bar{X}_2$
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Figure 3-3

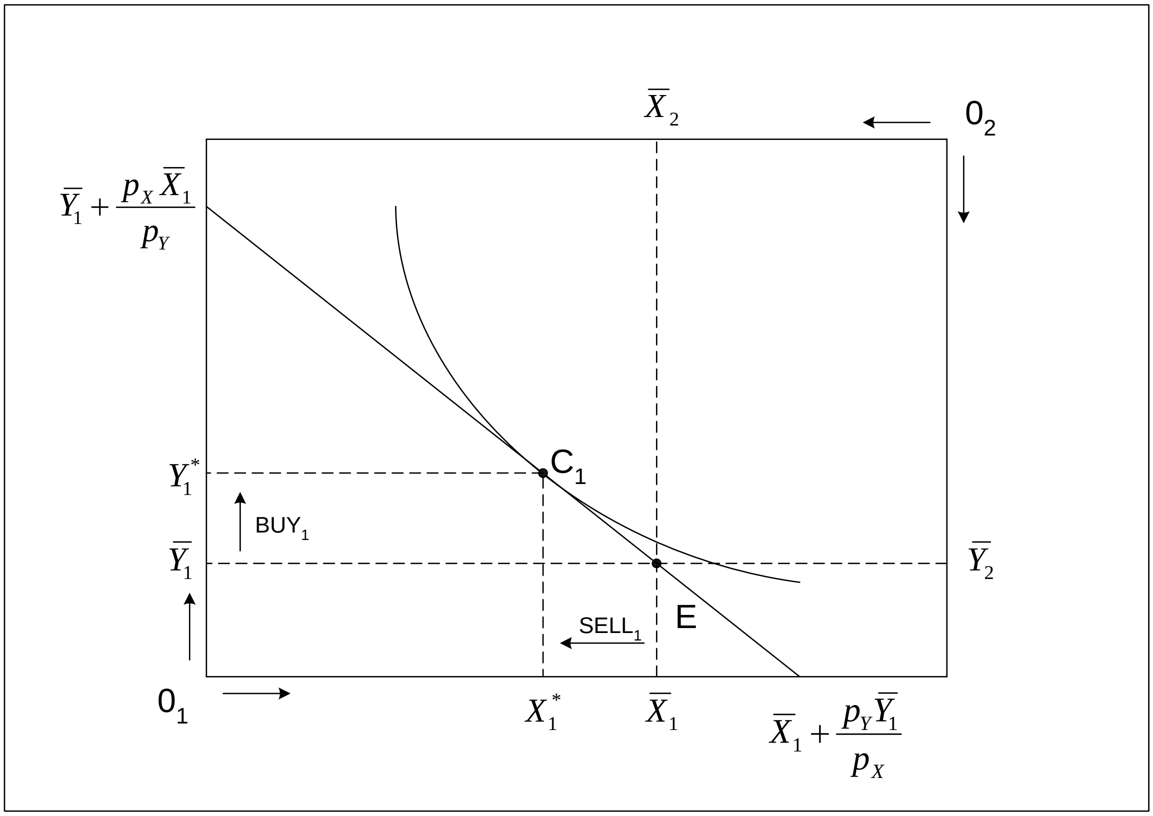
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Figure 3-4

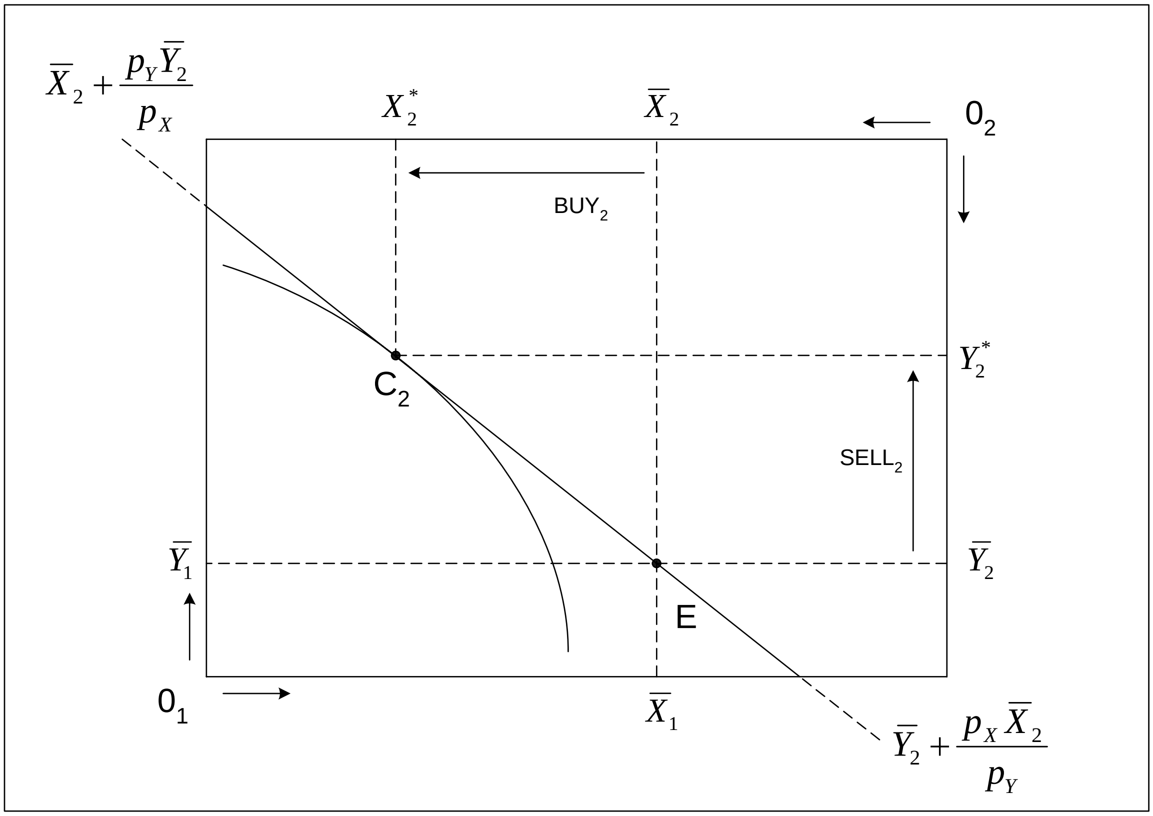
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Figure 3-5

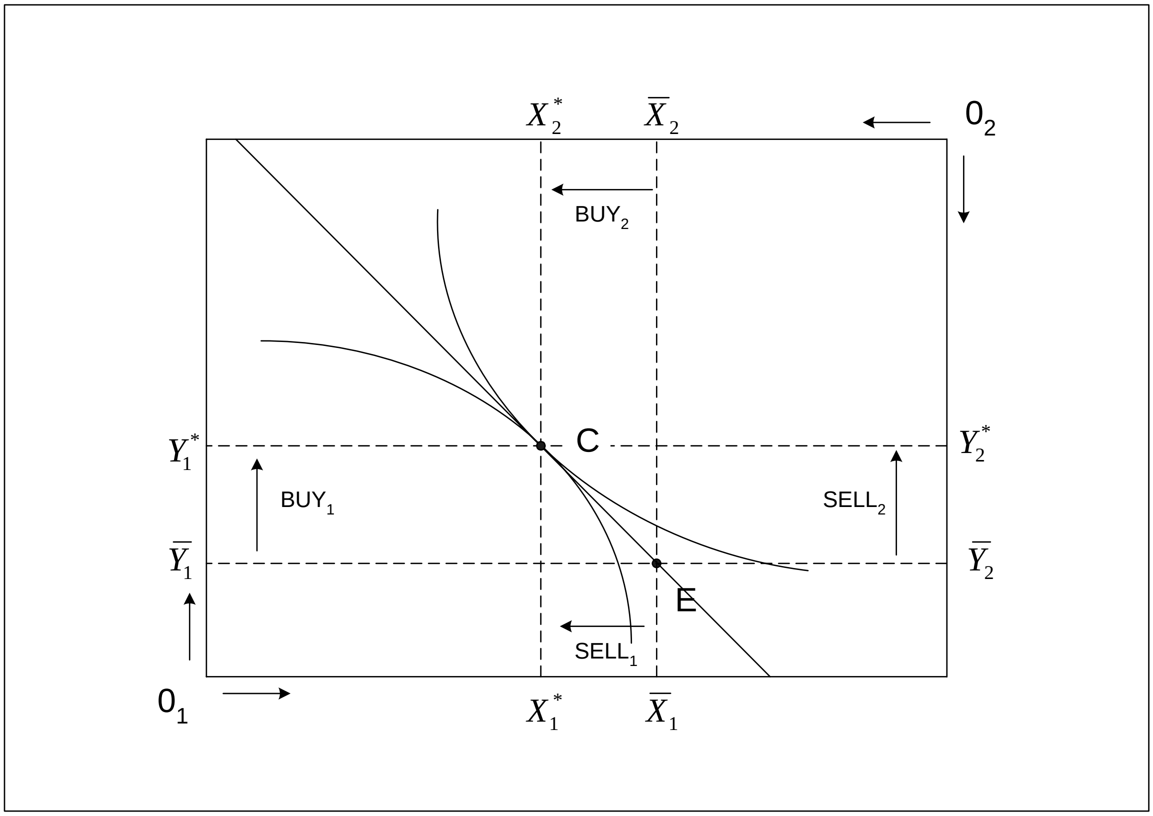
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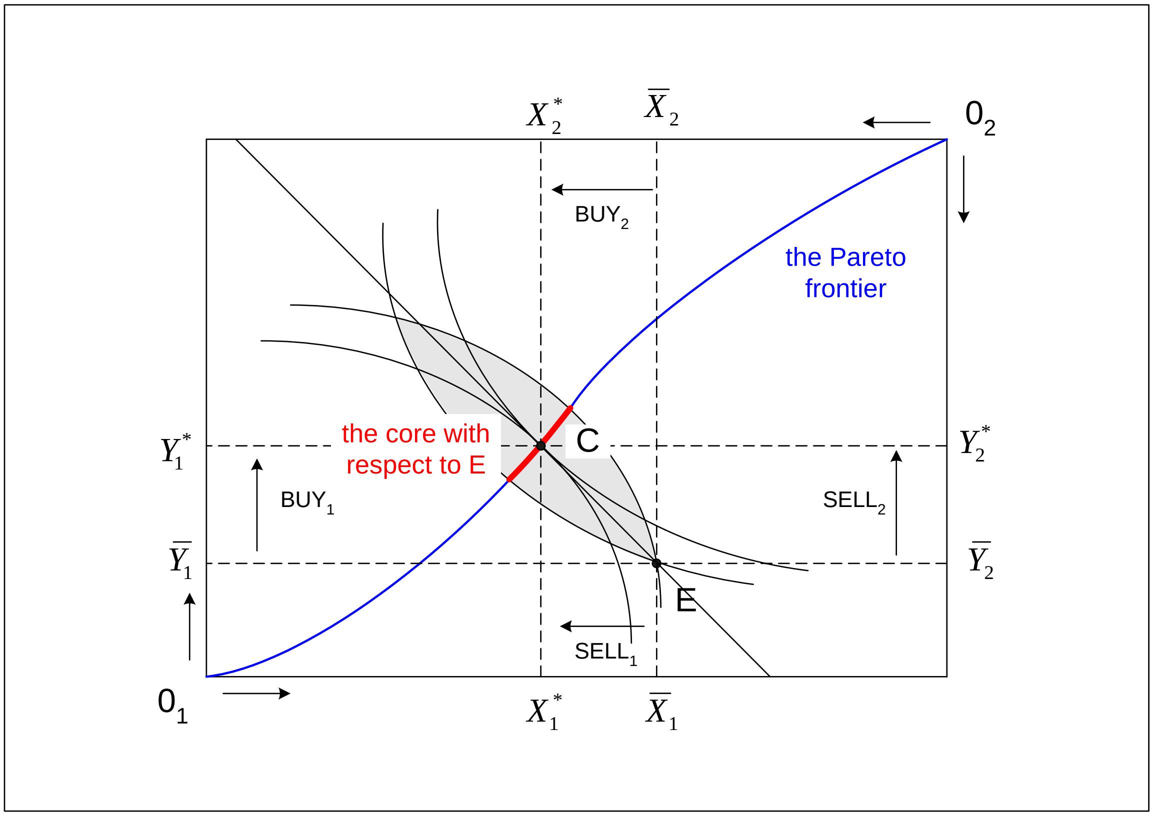
Figure 3-6



X_2^*
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BUY_2
the Pareto frontier
the core with respect to E
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Y_1^*
Y_2^*
BUY_1
$SELL_2$
$\bar{Y}_1$
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E
$SELL_1$
0_1
X_1^*
$\bar{X}_1$

Figure 3-7

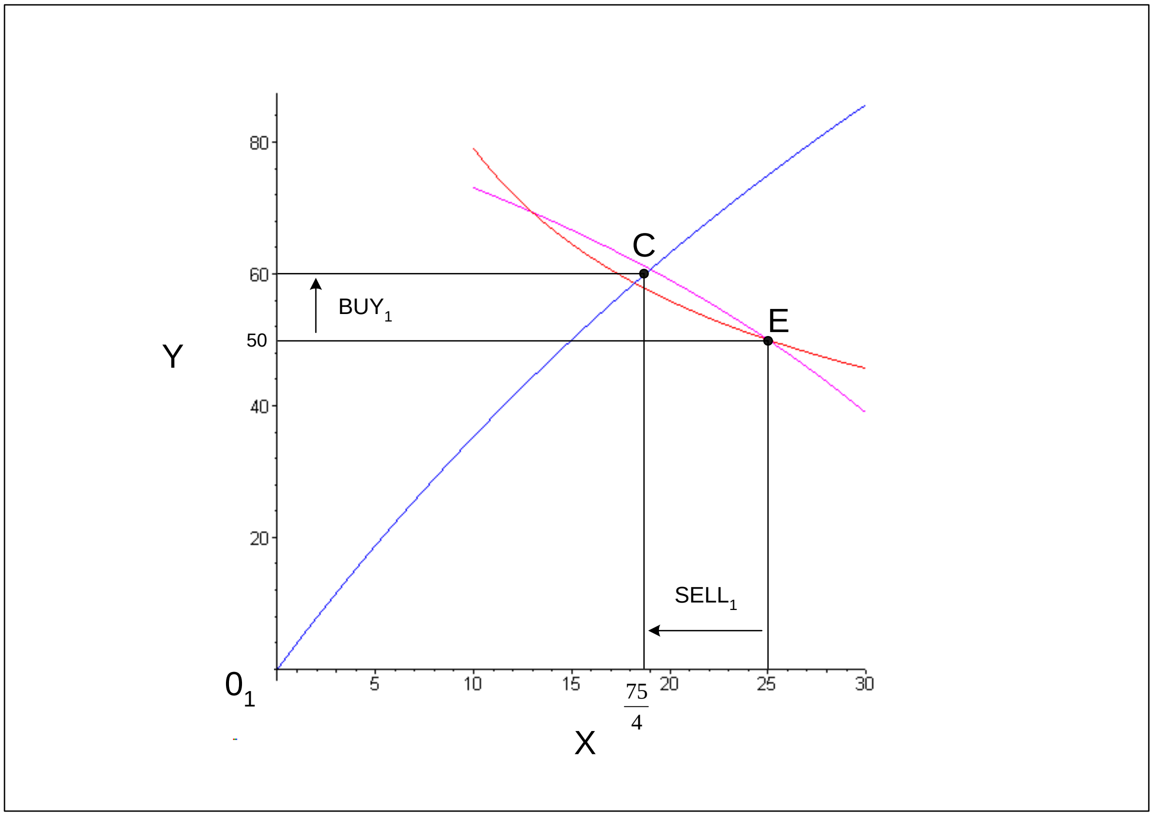

Figure 3-8