## 3.3 THE TWO-SECTOR GENERAL EQUILIBRIUM MODEL

There are two factors of production, called capital and labour ($K$ and $L$), two consumption goods ($x$ and $y$) produced by firms using $K$ and $L$, and $N$ individuals (consumers of $x$ and $y$ and suppliers of $K$ and $L$). There is a fixed amount of capital $\tilde{K}$ and a fixed amount of potential labour $\tilde{L}$ (some of which might be retained by individuals as leisure).

### 3.3-1 PRODUCTION

Firms in sector $X$ use production technology

$$X = f_X(K_X, L_X)$$

and firms in sector $Y$ use production technology

$$Y = f_Y(K_Y, L_Y)$$

We assume that both production functions exhibit *diminishing returns*: increasing the use of one input, while holding the other input fixed. causes output to rise but at a diminishing rate. See Figure 3-9.

All firms are price-takers in both input and output markets.

Profit for a representative firm in sector $j$ is the difference between revenue and cost:

$$\pi_j = p_j f_j(K_j, L_j) - wL_j - rK_j$$

where $w$ is the wage paid to labour, and $r$ is the rental rate paid to capital.

We assume that all firms act to maximize profit.

It turns out to be useful to think of profit-maximization as a two-stage problem:
(1) the firm chooses $K$ and $L$ to minimize the cost of producing a target level of output;
(2) the firm then chooses the target level of output to maximize profit.

This decomposition of the profit-maximization problem allows us to derive a cost function (from the first stage), from which we can then construct familiar concepts like marginal cost and average cost. It also allows us to deal with settings where the properties of the production function would make direct profit-maximization confusingly complicated; more on this later.

**Stage 1: Cost Minimization for a Firm in Sector Y**

The cost-minimization problem for a firm in sector $Y$ is

$$\min_{K_Y, L_Y} \quad wL_Y + rK_Y \quad \text{subject to} \quad f_Y(K_Y, L_Y) = y$$

where $y$ is the target level of output.

This is a *constrained-optimization problem* and we would typically solve it using the Lagrange method. However, we can also solve it by using some intuition derived from a graphical approach, as follows.

Figure 3-10 depicts an **isoquant** associated with the production function: it plots combinations and $K$ and $L$ that yield a given level of output, $y$. (The bowed-in shape reflects the diminishing returns to both factors).

Figure 3-10 also depicts a candidate input bundle that could be used to achieve that target output level, denoted $\{\hat{L}, \hat{K}\}$. A line with slope $-w/r$ passing through that candidate bundle represents the dollar cost of that bundle $\{\hat{L}, \hat{K}\}$; it is called an **isocost line**.

It is clear from Figure 3-10 that $\{\hat{L}, \hat{K}\}$ is <u>not</u> the cost-minimizing way to produce the target level of output. In particular, using less $L$ and more $K$ will reduce the cost of production until the firm reaches the input bundle labeled $\{L^*, K^*\}$ in Figure 3-11.

The defining characteristic of the bundle $\{L^*, K^*\}$ is that it occurs at a *tangency* between the isoquant and the isocost line passing through that point; that is, the slope of the isoquant is equal to the slope of the isocost line.

 

The absolute value of the slope of an isoquant is called the **technical rate of substitution** (TRS), and we can find it by total differentiation of the production function. For example, recall the production function in sector *Y*:

$$Y = f_Y(K_Y, L_Y)$$

Total differentiation yields

$$dY = \frac{\partial f_Y}{\partial K_Y} dK_Y + \frac{\partial f_Y}{\partial L_Y} dL_Y$$

As we move along an isoquant, changes in *K* and *L* leave output unchanged; that is, $dY = 0$ along the isoquant. Setting $dY = 0$ and solving for $dK_Y$ yields

$$dK_Y = -\left( \frac{\partial f_Y}{\partial L_Y} \Big/ \frac{\partial f_Y}{\partial K_Y} \right) dL_Y$$

Dividing both sides by $dL_Y$ then yields the slope of the isoquant:

$$\frac{dK_Y}{dL_Y} = -\left( \frac{\partial f_Y}{\partial L_Y} \Big/ \frac{\partial f_Y}{\partial K_Y} \right)$$

The TRS is defined as the absolute value of this slope:

$$TRS^Y \equiv \left| \frac{dK_Y}{dL_Y} \right| = \left( \frac{\partial f_Y}{\partial L_Y} \Big/ \frac{\partial f_Y}{\partial K_Y} \right)$$

The cost-minimizing solution occurs where the slope of the isoquant equals the slope of the isocost line; that is, where

(3.64) $$TRS^Y = \frac{w}{r}$$

See Figure 3-12.

What is the economic interpretation of (3.64)? If we rearrange (3.64) using the definition of the TRS, we have

$$\frac{\left(\dfrac{\partial f_Y}{\partial L_Y}\right)}{w} = \frac{\left(\dfrac{\partial f_Y}{\partial K_Y}\right)}{r}$$

The LHS measures the *marginal product* of *L* relative to its price. The RHS measures the marginal product of *K* relative to its price.

Suppose that these are not equal at the chosen input bundle. For example, suppose the input mix is such that

(3.65) $$\frac{\left(\dfrac{\partial f_Y}{\partial L_Y}\right)}{w} > \frac{\left(\dfrac{\partial f_Y}{\partial L_Y}\right)}{r}$$

At this input mix, input *L* provides greater "bang for the buck" than input *K*. Thus, the firm could reduce cost by switching away from *K* and into *L*. The converse would be true if the inequality in (3.65) is reversed. This switching should occur until we have equality.

Thus, if (3.64) is not satisfied then the firm could reduce cost by changing its input mix. Cost is only *minimized* if (3.64) holds.

We can now summarize the solution to the cost-minimization problem as a pair of equations:

(3.66) $$f_Y(K_y, L_y) = y$$

and

(3.67) $$TRS^Y = \frac{w}{r}$$

The first equation says that the production target is met; and the second equation tells us that the target is met at minimum cost.

We therefore have two equations in two unknowns $(K, L)$, and these can solved for the **conditional factor demands**, denoted $\hat{K}_Y(y,w,r)$ and $\hat{L}_Y(y,w,r)$. They are "conditional" demands in the sense that these factor demands are dependent on, or conditional on, the output target. See Figure 3.13.

The Cost Function

We can now construct the cost function for a firm in sector *Y*, denoted $c_Y(y,w,r)$.

By definition, the cost function tells us the minimum possible cost of producing a given level of output *y*. We have already seen that cost is minimized when inputs are chosen according to the conditional factor demands, so we simply need to calculate the cost of using those factor amounts in order to find the **cost function**:

$$c_Y(y,w,r) = w\hat{L}_Y(y,w,r) + r\hat{K}_Y(y,w,r)$$

We can then find marginal cost and average cost in the usual way:

$$MC_Y(y,w,r) = \frac{\partial c_Y(y,w,r)}{\partial y}$$

$$AC_Y(y,w,r) = \frac{c_Y(y,w,r)}{y}$$

An Example

Suppose the production function in sector *Y* is Cobb-Douglas:

$$Y = f_Y(K,L) = K^a L^b$$

where we have dropped the "*Y*" subscript on *K* and *L* in order to simplify notation.

The TRS for this production function is

$$TRS^Y = \left( \frac{\partial f_Y}{\partial L} \bigg/ \frac{\partial f_Y}{\partial K} \right) = \frac{bK^a L^{b-1}}{aK^{a-1}L^b} = \frac{bK}{aL}$$

Thus, the cost-minimization conditions are

(3.68) $$K^a L^b = y$$

and

(3.69) $$\frac{bK}{aL} = \frac{w}{r}$$

To solve these equations, rearrange (3.69) to obtain

(3.70) $$K = \frac{awL}{br}$$

and substitute (3.70) into (3.68) and solve for *L*:

(3.71) $$\hat{L}_Y(y,w,r) = y^{\frac{1}{a+b}} \left(\frac{br}{aw}\right)^{\frac{a}{a+b}}$$

This is the conditional demand for *L*. Note that it is increasing *y*, decreasing in *w*, and increasing in *r*.

We can now find the conditional demand for *K* by substituting (3.71) into (3.70):

(3.72) $$\hat{K}_Y(y,w,r) = y^{\frac{1}{a+b}} \left(\frac{aw}{br}\right)^{\frac{b}{a+b}}$$

Note that it is increasing *y*, decreasing in *r*, and increasing in *w*.

We can now find the cost function:

(3.73) $$\begin{aligned} c_Y(y,w,r) &= w\hat{L}_Y(y,w,r) + r\hat{K}_Y(y,w,r) \\ &= w\left( y^{\frac{1}{a+b}} \left(\frac{br}{aw}\right)^{\frac{a}{a+b}} \right) + r\left( y^{\frac{1}{a+b}} \left(\frac{aw}{br}\right)^{\frac{b}{a+b}} \right) \end{aligned}$$

After collecting terms, this reduces to a fairly simple expression:

(3.74) $$c_Y(y,w,r) = (yr^a w^b)^{\frac{1}{a+b}} \Omega$$

where

$$\Omega = \left( \left(\frac{b}{a}\right)^{\frac{a}{a+b}} + \left(\frac{a}{b}\right)^{\frac{b}{a+b}} \right)$$

is a summary term that is a function only of *a* and *b*.

Our primary interest is in the relationship between cost and output embodied in (3.74).

There are three distinct cases of interest:

(i) if $a+b<1$ then cost is increasing in output at an *increasing rate*; that is, MC is upward-sloping, and so AC is also upward-sloping. See Figures 3.14 and 3.15 (drawn for the case of $a+b=\frac{1}{2}$, which makes both functions linear).

(ii) if $a+b=1$ then cost is increasing in output at a *constant rate*; that is, MC is flat, and so AC is also flat (and equal to MC). See Figures 3.16 and 3.17.

(iii) if $a+b>1$ then cost is increasing in output at a *decreasing rate*; that is, MC is downward-sloping, and so AC is also downward-sloping. See Figures 3.18 and 3-19.

These three cases correspond to three different possibilities with respect to **returns to scale** embodied in the production function. Recall that returns to scale refers to what happens to output when all inputs are scaled up (for example, when all inputs are doubled).

In the Cobb-Douglas case, if we double the use of both *K* and *L* then output becomes

$$f_Y(2K,2L) = (2K)^a(2L)^b = 2^{a+b}K^aL^b = 2^{a+b}f_Y(K,L)$$

That is, output is scaled up by a factor $2^{a+b}$. Whether this is less than or greater than 2 depends on the size of $a+b$.

In particular, we have three cases:

 

(i) if $a+b<1$ then the production function exhibits **decreasing returns to scale** (DRS): output is less-than-doubled when all inputs are doubled.

(ii) if $a+b=1$ then the production function exhibits **constant returns to scale** (CRS): output is exactly doubled when all inputs are doubled.

(iii) if $a+b>1$ then the production function exhibits **increasing returns to scale** (IRS): output is more-than-doubled when all inputs are doubled.

The correspondence between returns to scale and the properties of the cost function is straightforward. Under CRS, if the firm wants to double its output then it needs to double its inputs, and that will double its cost; thus, the cost function will be linear.

Under DRS, if the firm wants to double its output then it must more-than-double its inputs, and so its cost will more-than-double. The converse is true if the production function exhibits IRS.

In practice, some inputs cannot be scaled up easily. Perhaps most importantly, managerial attention cannot be scaled up simply by adding more managers because communication between those managers becomes increasingly difficult as their numbers grow. In real economies, limited managerial attention is perhaps the single biggest constraint on firm size.

The simplest way to incorporate a non-scalable input into our model is to assume that every firm requires a fixed managerial-labour input (denoted $F_Y$ in sector Y) in addition to the labour used in production. This is a **quasi-fixed input**: it must be used if the firm is to produce any positive level of output but it is not needed if the firm produces nothing at all.

If we assume that each unit of managerial labour is paid wage *w*, then the cost function from (3.74) is modified to become

 

(3.75) $$c_Y(y,w,r) = (yr^a w^b)^{\frac{1}{a+b}}\Omega + wF_Y$$

The new cost term $wF_Y$ is a called a **quasi-fixed cost**: it is independent of output unless output is zero, in which case it is also zero. (Note that quasi-fixed costs exist at positive output even in the "long run").

Adding the quasi-fixed cost has no impact on marginal cost but it does change the average cost function in an important way. In particular, AC is now

(3.76) $$AC_Y(y,w,r) = \frac{c_Y(y,w,r)}{y} = \frac{(yr^a w^b)^{\frac{1}{a+b}}\Omega}{y} + \frac{wF_Y}{y}$$

The first RHS term is **average variable cost** (AVC); the second RHS term is **average fixed cost** (AFC).

The presence of the AFC has the potential to make the AC function U-shaped. In particular, if the production function exhibits DRS then AC will now be as depicted in Figure 3.20 (again drawn for the case where $a+b=\frac{1}{2}$, which makes MC linear).

Note that the MC curve must cross AC at its minimum point. The output level at this point is called **minimum efficient scale** (MES).

The term "efficient" here is used in an engineering sense and means only that AC is minimized at this point; it does <u>not</u> mean "Pareto efficient" in this context.

The case where MC is rising (but not necessarily linear) and AC is U-shaped is the most commonly assumed cost structure in models of the economy. We will assume that same case throughout our analysis here.[1]

**Stage 2: Profit Maximization for a Firm in Sector Y**

The profit-maximization problem for a firm in sector *Y* is

$$\max_{y} \quad p_Y y - c_Y(y, w, r)$$

That is, the firm chooses its level of output to maximize the difference between revenue and cost.

This is an unconstrained-optimization problem and we solve it by taking the derivate with respect to *y* and setting that derivate equal to zero:

$$p_Y - \frac{\partial c_Y(y, w, r)}{\partial y} = 0$$

The second LHS term is marginal cost. Thus, the firm chooses output so that its marginal cost at that output is just equal to the output price:

$$MC_Y(y, w, r) = p_Y$$

The solution to this profit-maximization condition yields the **supply function** for this firm, denoted $y(p_Y, w, r)$. The solution is depicted in Figure 3-21 (drawn for the case where MC is linear).

[1] In cases where the MC is not upward-sloping (as in the CRS and IRS cases described earlier), there is no simple solution to the profit-maximization problem. The IRS case may not even be compatible with price-taking behaviour at all; we may instead have “natural monopoly”.

<u>Example</u>

Recall the cost function from the Cobb-Douglas example:

$$c_Y(y,w,r) = (yr^a w^b)^{\frac{1}{a+b}}\Omega + wF_Y$$

The marginal cost function is

$$MC_Y(y,w,r) = \frac{\partial c_Y(y,w,r)}{\partial y} = \frac{y^{\frac{1}{a+b}-1}(r^a w^b)^{\frac{1}{a+b}}\Omega}{a+b}$$

Setting $MC_Y(y,w,r) = p_Y$ and solving for *y* yields the supply function

(3.77) $$y(p_Y, w, r) = \left(\frac{{p_Y}^{a+b}}{r^a w^b}\right)^{\frac{1}{1-a-b}}\Psi$$

where

$$\Psi = \left(\frac{a+b}{\Omega}\right)^{\frac{a+b}{1-a-b}}$$

is a summary term that is a function only of *a* and *b*.

This supply function is increasing in $p_Y$, and decreasing in both *w* and *r*. That is, output rises as the output price rises, but output falls as input prices rise.

Note that the relationship between supply and the price of an input depends specifically on the importance of that input in the production function (as reflected in the parameters *a* and *b*).

**The Factor Demands**

Recall that the cost-minimization problem yields the conditional factor demands: $\hat{K}_Y(y,w,r)$ and $\hat{L}_Y(y,w,r)$. These are conditional on a particular level of output.

 

If we now set that conditional output level equal to the profit-maximizing output then we have the (unconditional) **factor demands**:

$$K_Y(p_Y, w, r) = \hat{K}_Y(y(p_Y, w, r), w, r)$$

$$L_Y(p_Y, w, r) = \hat{L}_Y(y(p_Y, w, r), w, r)$$

That is, we construct the factor demands by substituting the supply function for *y* in the conditional factor demands.

<u>Example</u>

Recall the conditional factor demands from the Cobb-Douglas example (before adding the fixed managerial-labour requirement):

$$\hat{K}_Y(y, w, r) = y^{\frac{1}{a+b}}\left(\frac{aw}{br}\right)^{\frac{b}{a+b}} \quad \text{and} \quad \hat{L}_Y(y, w, r) = y^{\frac{1}{a+b}}\left(\frac{br}{aw}\right)^{\frac{a}{a+b}}$$

If we substitute the supply function from (3.77) for *y* in the conditional factor demands then we obtain the factor demands, and after collecting terms, these can be expressed as

$$K_Y(p_Y, w, r) = \left(\frac{p_Y}{w^b r^{\left(\frac{a+b(1-a-b)}{a+b}\right)}}\right)^{\frac{1}{1-a-b}} \Phi_K$$

where

$$\Phi_K = \left(\frac{a+b}{\Omega}\right)^{\frac{1}{1-a-b}}\left(\frac{a}{b}\right)^{\frac{b}{a+b}}$$

is a summary term that is a function only of *a* and *b*, and

$$L_Y(p_Y, w, r) = \left(\frac{p_Y}{r^a w^{\left(\frac{b+a(1-a-b)}{a+b}\right)}}\right)^{\frac{1}{1-a-b}} \Phi_L$$

where

 

$$\Phi_L = \left(\frac{a+b}{\Omega}\right)^{\frac{1}{1-a-b}} \left(\frac{b}{a}\right)^{\frac{a}{a+b}}$$

is another summary term that is a function only of *a* and *b*.

These factor demands exhibit properties that we might expect: they are both increasing in the output price, and decreasing in their own prices.

Perhaps more interesting are the cross-price effects: the impact of a rise in *w* on the demand for *K*; and the impact of a rise in *r* on the demand for *L*.

The cross-price effect in each case can be decomposed into a **substitution effect** and an **output effect**. An increase in one factor price causes the firm to substitute into the other factor (a positive effect) but it also causes the firm to reduce output overall and so demand less of both factors (a negative effect). For the Cobb-Douglas example, the output effect always outweighs the substitution effect, and so the firm demands less of both factors when the price of either factor rises.

Note that $L_Y(p_Y, w, r)$ is the labour used only in production; we have not included the quasi-fixed managerial-labour requirement yet, and so we will need to add that later when calculating the total amount of labour hired by the firm.

## The Profit Function

Recall that profit for a firm in sector Y is

$$p_Y y - c_Y(y, w, r)$$

We can now evaluate this profit at the profit-maximizing choice of *y* to find the **profit function** for this firm:

$$\pi_Y(p_Y, r, w) = p_Y y(p_Y, r, w) - c_Y(y(p_Y, r, w), w, r)$$

This function tells us the value of *maximized* profit, as a function of prices.

Example

Recall that for the Cobb-Douglas example, the cost function is given by

$$c_Y(y, w, r) = (yr^a w^b)^{\frac{1}{a+b}} \Omega + wF_Y$$

and the supply function is given by

$$y(p_Y, w, r) = \left( \frac{p_Y{}^{a+b}}{r^a w^b} \right)^{\frac{1}{1-a-b}} \Psi$$

Making these substitutions into the expression for profit yields an expression for maximized profit as a function of prices, which after some simplification of terms, is given by

$$\pi(p_Y, w, r) = \left( \frac{p_Y}{r^a w^b} \right)^{\frac{1}{1-a-b}} (1 - a_1 - b_1)\Psi - wF_Y$$

**Aggregate Supply in Sector Y**

Aggregate supply in sector Y is the sum of all the individual-firm supplies. Since all firms in the sector use the same technology (by assumption), we can construct the aggregate supply function as

$$S_Y(p_Y, w, r) = n_Y y(p_Y, w, r)$$

where $n_Y$ is the number of firms in sector Y.

**Aggregate Factor Demands in Sector Y**

The aggregate demand for labour in sector Y is the sum of all the individual-firm demands for labour. Since all firms in the sector use the same technology, we can construct that aggregate demand for labour in sector Y as

 

$$D_L^Y(p_Y, w, r) = n_Y L_Y(p_Y, w, r) + n_Y F_Y$$

where we have now added the quasi-fixed managerial-labour requirements.

Similarly, aggregate demand for capital in sector Y is

$$D_K^Y(p_Y, w, r) = n_Y K_Y(p_Y, w, r)$$

**Aggregate Profit in Sector Y**

Aggregate profit in sector Y is the sum of all the individual-firm profits. Since all firms in the sector use the same technology, we can construct that aggregate profit as

$$\Pi_Y(p_Y, w, r) = n_Y \pi_Y(p_Y, w, r)$$

**Supply and Factor Demands in Sector X**

We have so far examined only the behaviour of firms in sector Y. We now need to do the same for firms in sector X. Firms in sector X behave in the same way as firms in sector Y; they choose output and factor inputs to maximize profit.

We will not go through the analysis of this sector in the same descriptive detail as we did for sector Y; we will instead just summarize the steps that we take to calculate the supply function and the factor demands.

**Cost Minimization for a Firm in Sector X**

The cost-minimization problem for a firm in sector X is

$$\min_{K_X, L_X} \quad wL_X + rK_X \quad \text{subject to} \quad f_X(K_X, L_X) = x$$

where $x$ is the target level of output.

The cost-minimizing solution occurs where the slope of the isoquant equals the slope of the isocost line (the tangency condition):

 

$$TRS^X = \frac{w}{r}$$

where

$$TRS^X \equiv \left|\frac{dK_X}{dL_X}\right| = \left( \frac{\partial f_X}{\partial L_X} \Big/ \frac{\partial f_X}{\partial K_X} \right)$$

Using the tangency condition combined with the output constraint, we can solve for the conditional factor demands $\hat{K}_X(x, w, r)$ and $\hat{L}_X(x, w, r)$.

We can then construct the cost function

$$c_X(x, w, r) = w\hat{L}_X(x, w, r) + r\hat{K}_X(x, w, r) + wF_X$$

where $F_X$ is the quasi-fixed input of managerial labour in a sector-X firm.

Marginal cost is

$$MC_X(x, w, r) = \frac{\partial c_X(x, w, r)}{\partial x}$$

and average cost is

$$AC_X(x, w, r) = \frac{c_X(x, w, r)}{x}$$

**Profit Maximization for a Firm in Sector X**

The profit-maximization problem for a firm in sector X is

$$\max_{x} \quad p_X x - c_X(x, w, r)$$

and we solve it by taking the derivate with respect to *x* and setting that derivate equal to zero:

$$p_X - \frac{\partial c_X(x, w, r)}{\partial x} = 0$$

 

Thus, the firm chooses output so that its marginal cost at that output is just equal to the output price:

$$MC_X(x, w, r) = p_X$$

The solution to this profit-maximization condition yields the supply function for this firm, denoted $x(p_X, w, r)$.

The profit function for a firm in sector X is constructed as

$$\pi_X(p_X, r, w) = p_X x(p_X, r, w) - c_X(x(p_X, r, w), w, r)$$

Recall that this function tells us the value of *maximized* profit, as a function of prices.

**The Factor Demands in Sector X**

We construct the factor demands in sector X by substituting the supply function for *x* in the conditional factor demands.

$$K_X(p_X, w, r) = \hat{K}_X(x(p_X, w, r), w, r)$$

$$L_X(p_X, w, r) = \hat{L}_X(x(p_X, w, r), w, r)$$

**Aggregate Supply in Sector X**

Since all firms in the sector use the same technology (by assumption), we can construct aggregate supply in sector X as

$$X(p_X, w, r) = n_X x(p_X, w, r)$$

where $n_X$ is the number of firms in sector X.

**Aggregate Factor Demands in Sector X**

Since all firms in the sector use the same technology, we can construct the aggregate demand for labour in sector X as

$$D_L^X(p_X, w, r) = n_X L_X(p_X, w, r) + n_X F_X$$

where $n_X F_X$ is the aggregate managerial-labour requirement in the sector.

Similarly, aggregate demand for capital in sector X is

$$D_K^X(p_X, w, r) = n_X K_X(p_X, w, r)$$

**Aggregate Profit in Sector X**

Aggregate profit in sector X is the sum of all the individual-firm profits. Since all firms in the sector use the same technology, we can construct that aggregate profit as

$$\Pi_X(p_X, w, r) = n_X \pi_X(p_X, w, r)$$

**3.3-2 PRODUCTION: A NUMERICAL EXAMPLE**

Sector Y

Suppose the production technology in sector Y is

$$Y = f_Y(K, L) = K^{\frac{1}{4}} L^{\frac{1}{4}}$$

and that there is a quasi-fixed managerial-labour input requirement $F_Y$.

First find the TRS for this production function

$$TRS^Y \equiv \left| \frac{dK}{dL} \right| = \left( \frac{\partial f_Y}{\partial L} \Big/ \frac{\partial f_Y}{\partial K} \right) = \left( \frac{\frac{1}{4} K^{\frac{1}{4}} L^{-\frac{3}{4}}}{\frac{1}{4} K^{-\frac{3}{4}} L^{\frac{1}{4}}} \right) = \frac{K}{L}$$

Cost minimization requires that this TRS is equated to the factor price ratio (the tangency condition):

$$\frac{K}{L} = \frac{w}{r}$$

and also that the output target is met:

$$K^{\frac{1}{4}}L^{\frac{1}{4}} = y$$

Substitute the first equation into the second to yield:

$$\left(\frac{wL}{r}\right)^{\frac{1}{4}}L^{\frac{1}{4}} = y$$

and solve this for *L* to yield the conditional demand for labour:

$$\hat{L}_Y(y,w,r) = y^2\left(\frac{r}{w}\right)^{\frac{1}{2}}$$

Now substitute this back into the tangency condition and solve for *K*:

$$\hat{K}_Y(y,w,r) = y^2\left(\frac{w}{r}\right)^{\frac{1}{2}}$$

This is the conditional demand for capital.

Construct the cost function from the conditional factor demands:

$$c_Y(y,w,r) = w\hat{L}_Y(y,w,r) + r\hat{K}_Y(y,w,r) + wF_Y = 2y^2(wr)^{\frac{1}{2}} + wF_Y$$

Marginal cost is then given by

$$MC_Y(y,w,r) = \frac{\partial c_Y(y,w,r)}{\partial y} = 4y(wr)^{\frac{1}{2}}$$

and average cost is

$$AC_Y(y,w,r) = \frac{c_Y(y,w,r)}{y} = 2y(wr)^{\frac{1}{2}} + \frac{wF_Y}{y}$$

Note that in this example, MC is linear in output. That reflects the fact that $a + b = \frac{1}{2}$ in this example.

Profit maximization requires that MC is equated to price:

$$4y(wr)^{\frac{1}{2}} = p_Y$$

Solve this condition to yield the supply function:

$$y(p_Y, w, r) = \frac{p_Y}{4(wr)^{\frac{1}{2}}}$$

Thus, the supply function in this example is linear in the output price (because MC is linear in output; see Figure 3-21).

Substitute the supply function into the conditional factor demands to find the factor demands:

$$L_Y(p_Y, w, r) = \hat{L}_Y(y(p_Y, w, r), w, r) = \left(\frac{p_Y}{4w^{\frac{1}{2}}r^{\frac{1}{2}}}\right)^2 \left(\frac{r}{w}\right)^{\frac{1}{2}} = \frac{p_Y^2}{16w^{\frac{3}{2}}r^{\frac{1}{2}}}$$

$$K_Y(p_Y, w, r) = \hat{K}_Y(y(p_Y, w, r), w, r) = \left(\frac{p_Y}{4w^{\frac{1}{2}}r^{\frac{1}{2}}}\right)^2 \left(\frac{w}{r}\right)^{\frac{1}{2}} = \frac{p_Y^2}{16w^{\frac{1}{2}}r^{\frac{3}{2}}}$$

Substitute the supply function into the definition of profit to find the profit function:

$$\pi_Y(p_Y, w, r) = p_Y y(p_Y, w, r) - c_Y(y(p_Y, w, r), w, r)$$

$$= p_Y\left(\frac{p_Y}{4(wr)^{\frac{1}{2}}}\right) - 2\left(\frac{p_Y}{4(wr)^{\frac{1}{2}}}\right)^2 (wr)^{\frac{1}{2}} - wF_Y$$

$$= \frac{p_Y^2}{8(wr)^{\frac{1}{2}}} - wF_Y$$

 

The aggregate supply function in sector Y is

$$S_Y(p_Y, w, r) = n_Y y(p_Y, w, r) = n_Y \left( \frac{p_Y}{4(wr)^{\frac{1}{2}}} \right)$$

and the aggregate factor demands in this sector are

$$D_L^Y(p_Y, w, r) = n_Y L_Y(p_Y, w, r) = n_Y \left( \frac{p_Y^2}{16 w^{\frac{3}{2}} r^{\frac{1}{2}}} \right) + n_Y F_Y$$

and

$$D_K^Y(p_Y, w, r) = n_Y K_Y(p_Y, w, r) = n_Y \left( \frac{p_Y^2}{16 w^{\frac{1}{2}} r^{\frac{3}{2}}} \right)$$

where $n_Y$ is the number of firms in sector Y.

Aggregate profit in sector Y is

$$\Pi_Y(p_Y, w, r) = n_Y \left( \frac{p_Y^2}{8(wr)^{\frac{1}{2}}} - wF_Y \right)$$

<u>Sector X</u>

Suppose the production technology in sector X is

$$X = f_X(K, L) = K^{\frac{1}{4}} L^{\frac{1}{2}}$$

and that there is a quasi-fixed managerial-labour input requirement of $F_X$.

First find the TRS for this production function

$$TRS^X \equiv \left| \frac{dK}{dL} \right| = \left( \frac{\partial f_X}{\partial L} \Big/ \frac{\partial f_X}{\partial K} \right) = \left( \frac{\frac{1}{2} K^{\frac{1}{4}} L^{-\frac{1}{2}}}{\frac{1}{4} K^{-\frac{3}{4}} L^{\frac{1}{2}}} \right) = \frac{2K}{L}$$

Cost minimization requires that the TRS is equated to the factor price ratio (the tangency condition):

$$\frac{2K}{L} = \frac{w}{r}$$

and that the output target is met:

$$K^{\frac{1}{4}} L^{\frac{1}{2}} = x$$

Substitute the first equation into the second to yield:

$$\left(\frac{wL}{2r}\right)^{\frac{1}{4}} L^{\frac{1}{2}} = x$$

and solve this for $L$ to yield the conditional demand for labour:

$$\hat{L}_X(x, w, r) = x^{\frac{4}{3}} \left(\frac{2r}{w}\right)^{\frac{1}{3}}$$

Now substitute this back into the tangency condition and solve for $K$:

$$\hat{K}_X(x, w, r) = x^{\frac{4}{3}} \left(\frac{w}{2r}\right)^{\frac{2}{3}}$$

This is the conditional demand for capital.

Construct the cost function from the conditional factor demands:

$$c_X(x, w, r) = w\hat{L}_X(x, w, r) + r\hat{K}_X(x, w, r) + wF_X \; = 3x^{\frac{4}{3}} \left(\frac{w}{2}\right)^{\frac{2}{3}} r^{\frac{1}{3}} + wF_X$$

Marginal cost is

$$MC_X(x, w, r) = \frac{\partial c_X(x, w, r)}{\partial x} = 4x^{\frac{1}{3}} \left(\frac{w}{2}\right)^{\frac{2}{3}} r^{\frac{1}{3}}$$

and average cost is

$$AC_X(x, w, r) = \frac{c_X(x, w, r)}{x} = 3x^{\frac{1}{3}} \left(\frac{w}{2}\right)^{\frac{2}{3}} r^{\frac{1}{3}} + \frac{wF_X}{x}$$

 

Profit maximization requires that MC is equated to price:

$$4x^{\frac{1}{3}}\left(\frac{w}{2}\right)^{\frac{2}{3}} r^{\frac{1}{3}} = p_X$$

Solve this condition to yield the supply function:

$$x(p_X, w, r) = \frac{p_X^3}{16w^2 r}$$

Note that this supply function – unlike the one for a firm in sector Y – is <u>not</u> linear in price (because $a + b > \frac{1}{2}$, which makes MC strictly concave; see Figure 3-22).

Substitute the supply function into the conditional factor demands to find the factor demands:

$$L_X(p_X, w, r) = \hat{L}_X(x(p_X, w, r), w, r) = \left(\frac{p_X^3}{16w^2 r}\right)^{\frac{4}{3}}\left(\frac{2r}{w}\right)^{\frac{1}{3}} = \frac{p_X^4}{32w^3 r}$$

$$K_X(p_X, w, r) = \hat{K}_X(x(p_X, w, r), w, r) = \left(\frac{p_X^3}{16w^2 r}\right)^{\frac{4}{3}}\left(\frac{w}{2r}\right)^{\frac{2}{3}} = \frac{p_X^4}{64w^2 r^2}$$

Substitute the supply function into the definition of profit to find the profit function:

$$\pi_X(p_X, r, w) = p_X x(p_X, r, w) - c_X(x(p_X, r, w), w, r)$$

$$= p_X\left(\frac{p_X^3}{16w^2 r}\right) - 3\left(\frac{p_X^3}{16w^2 r}\right)^{\frac{4}{3}}\left(\frac{w}{2}\right)^{\frac{2}{3}} r^{\frac{1}{2}} - wF_X$$

$$= \frac{p_X^4}{64w^2 r} - wF_X$$

The aggregate supply function in sector X is

$$S_X(p_X, w, r) = n_X x(p_X, w, r) = n_X\left(\frac{p_X^3}{16w^2 r}\right)$$

and the aggregate factor demands in this sector are

$$D_L^X(p_X, w, r) = n_X L_X(p_X w, r) = n_X\left(\frac{p_X^4}{32w^3 r}\right) + n_X F_X$$

and

$$D_K^X(w, r, p_X) = n_X K_X(p_X, w, r) = n_X\left(\frac{p_X^4}{64w^2 r^2}\right)$$

where $n_X$ is the number of firms in sector X.

Aggregate profit in sector X is

$$\Pi_X(p_X, w, r) = n_X\left(\frac{p_X^4}{64w^2 r} - wF_X\right)$$

**3.3-3 CONSUMPTION**

There are *N* individuals (persons) , and person *i* is endowed with a fixed amount of capital $\overline{K}_i$ and a fixed amount of potential labour $\overline{L}_i$ which can be split between leisure and actual labour supplied to the market.

The aggregation of these individual factor endowments yield the total amounts available in the economy as a whole:

$$\tilde{K} = \sum_{i=1}^{N} \overline{K}_i$$

and

$$\tilde{L} = \sum_{i=1}^{N} \overline{L}_i$$

Individuals are also ultimately the owners of firms, either directly or via shares held in mutual funds and other financial instruments. This means that any profits made by firms ultimately accrue to individuals. We will later assume that free entry drives profits to zero in equilibrium, and so there will be no profits that accrue to individuals.

 

However, for the moment we will allow for the possibility of positive profits. Recall that these profits are function of prices, and denoted $\Pi_Y(p_Y, w, r)$ and $\Pi_X(p_X, w, r)$ for aggregate profits in sectors Y and X respectively.

We assume that person *i* is endowed with a firm-ownership mix that provides her with profit-based income equal to

$$d_i(p_X, p_Y, w, r) = \theta_{iX}\Pi_X(p_X, w, r) + \theta_{iY}\Pi_Y(p_Y, w, r)$$

where $\theta_{iX}$ is her fractional claim on profits in sector X, and $\theta_{iY}$ is her fractional claim on profits in sector Y. We can think of $d_i$ as her dividend income. Note that this income is a function of prices because profits are a function of prices.

We can now define the **wealth** of person *i* as the market value of her endowment, including her claim over profits:

$$M_i(p_X, p_Y, w, r) = r\overline{K}_i + w\overline{L}_i + d_i(p_X, p_Y, w, r)$$

It might be tempting to think that $\overline{L}_i$ is the same for all people – since everyone has 24 hours in their day – but we know that some people are endowed with *more productive* labour than others (due to differences in genetic make-up and differences in upbringing). Thus, some people effectively have greater labour endowments than others (and this accounts for at least some of the inequality typically observed in actual market outcomes).

In practice, individuals often form *households*, and make consumption and labour-supply decisions collectively. This is an important property of real economies but we abstract from it here in order to keep the model simple. Thus, we assume that people act as individuals.

Each person has preferences defined over their consumption of the two goods, and over the amount of leisure enjoyed, denoted $l$. Those preferences are represented by a utility function $u_i(x_i, y_i, l_i)$.

**The Utility-Maximization Problem**

The decision problem for person *i* is to maximize her utility subject to her wealth constraint:

$$\max_{x,y,l} \; u_i(x_i, y_i, l_i) \quad \text{subject to} \quad p_X x_i + p_Y y_i = M_i - wl_i$$

where $wl_i$ is the amount of potential labour income given up because of time spent in leisure. (Note that we have suppressed the functional dependence of $M_i$ on prices to reduce notational clutter).

If we rearrange the constraint, we can interpret the problem as one where the individual uses her wealth to purchase three goods, where the third good is leisure and where its "price" is *w*:

$$\max_{x,y,l} \; u_i(x_i, y_i, l_i) \quad \text{subject to} \quad p_X x_i + p_Y y_i + wl_i = M_i$$

This is a *constrained-optimization problem* and we would typically solve it using the Lagrange method. However, we can also solve it using some intuition derived from a graphical approach, as follows.

The problem is fundamentally no different from the familiar consumption-choice problem over two goods, represented graphically in a diagram with indifference curves and a budget constraint, except now we need three dimensions to represent it.

Consider Figure 3-23. It depicts an **indifference surface** where *x* and *y* are measured on the horizontal axes and *l* is measured on the vertical axis. By definition, any point on this surface yields the same utility as any other point on the surface.

The 3D representation of the wealth constraint is a **plane**. All points on the plane have the same total expenditure as all other points on the plane. See Figure 3-24.

The solution to the utility-maximization problem is a tangency between the wealth-constraint plane and the indifference surface in 3D space; see Figure 3-25.

We can think of this tangency in 3D space as comprising two parts: a "horizontal tangency" viewed from the top, looking down; and a "vertical tangency" viewed from side-on (see Figures 3-26 for a depiction of the latter).

The "horizontal tangency" is the familiar one from the standard 2D problem:

$$MRS_{xy}^{i} = \frac{p_X}{p_Y}$$

That is, the marginal rate of substitution between $x$ and $y$ is equal to the ratio of prices for $x$ and $y$.

The "vertical tangency" sets the marginal rate of substitution between $l$ and $y$ equal to the ratio of prices for $l$ and $y$:

$$MRS_{ly}^{i} = \frac{w}{p_Y}$$

These two tangency conditions together with the wealth constraint constitute the solution to the utility-maximization problem, and they can be solved to find the consumption choices, and the labour supply choice.

<u>Example</u>

Suppose preferences are Cobb-Douglas:

$$u(x, y, l) = x^{\alpha} y^{\beta} l^{\delta}$$

This is just a generalization of the utility function we saw in Section 3.2 where we now include a third good, leisure.

To find the MRS expressions for this function we first need to find its total derivative:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial l}dl$$
$$= (\alpha x^{\alpha-1}y^{\beta}l^{\delta})dx + (\beta x^{\alpha}y^{\beta-1}l^{\delta})dy + (\delta x^{\alpha}y^{\beta}l^{\delta-1})dl$$

By definition, utility is constant along an indifference surface so utility cannot change as we change *x*, *y* and *l*; that is, $du = 0$.

Now suppose we move horizontally along the indifference surface, which keeps *l* fixed at some level. That is, we set $dl = 0$ and just change *x* and *y*. Setting $du = 0$ and $dl = 0$, and then solving for $dy$ allows us to construct the MRS between *x* and *y*:

$$MRS_{xy} = \left|\frac{dy}{dx}\right| = \frac{\alpha x^{\alpha-1}y^{\beta}l^{\delta}}{\beta x^{\alpha}y^{\beta-1}l^{\delta}} = \frac{\alpha y}{\beta x}$$

Now suppose we move vertically along the indifference surface, keeping *x* fixed at some level. That is, we set $dx = 0$ and just change *l* and *y*. Setting $du = 0$ and $dx = 0$, and then solving for $dy$ allows us to construct the MRS between *l* and *y*:

$$MRS_{ly} = \left|\frac{dy}{dl}\right| = \frac{\delta x^{\alpha}y^{\beta}l^{\delta-1}}{\beta x^{\alpha}y^{\beta-1}l^{\delta}} = \frac{\delta y}{\beta l}$$

Thus, the MRS expressions for these Cobb-Douglas preferences are very simple (and are straightforward generalizations of the expressions we found in Section 3.2 for the case of two goods).

The utility-maximization conditions can now be summarized are

(3.78) $$\frac{\alpha y}{\beta x} = \frac{p_X}{p_Y}$$

(3.79) $$\frac{\delta y}{\beta l} = \frac{w}{p_Y}$$

(3.80) $$p_X x + p_Y y + wl = M$$

 

These equations can be solved by substitution. In particular, use (3.78) and (3.79) to express $x$ and $l$ in terms of $y$, and then substitute these into (3.80) and solve for $y$:

$$y(p_X, p_Y, w, r) = \frac{\beta M}{(\alpha+\beta+\delta)p_Y} \quad (3.81)$$

Now substitute this solution for $y$ in (3.78) and solve for $x$, and in (3.79) and solve for $l$:

$$x(p_X, p_Y, w, r) = \frac{\alpha M}{(\alpha+\beta+\delta)p_X} \quad (3.82)$$

$$l(p_X, p_Y, w, r) = \frac{\delta M}{(\alpha+\beta+\delta)w} \quad (3.83)$$

We can now use (3.83) to construct labour supply for this individual:

$$L(p_X, p_Y, w, r) = \overline{L} - l(p_X, p_Y, w) = \overline{L} - \frac{\delta M}{(\alpha+\beta+\delta)w} \quad (3.84)$$

Note that the cross-price effects in these consumption choices are all zero; only the own-price matters. This is an artifact of the Cobb-Douglas structure, and it is an implausible property in practice. However, it will simply the algebra in this example.

Note too that wealth ($M$) enters all of these expressions linearly. This is also an artifact of the Cobb-Douglas structure, and it too will make things easier for us later. In particular, it will allow us to express aggregate demand functions in terms of aggregate wealth if we impose some additional assumptions.

**Aggregate Demands and Aggregate Labour Supply**

The consumption choices in (3.81) – (3.83) are for a particular individual, and we know that preferences and wealth vary across individuals. Thus, we cannot find aggregate values just by multiplying by the number of individuals. (We did that on the production side but in that setting it is not unreasonable to assume that all firms have access to the same technology).

 

To find aggregate values we must sum across individual values. In particular, let $y_i(p_X, p_Y, w, r)$ denote the demand for good *y* by individual *i*. Then the aggregate demand for good *y* is

$$D_Y(p_X, p_Y, w, r) = \sum_{i=1}^{N} y_i(p_X, p_Y, w, r)$$

Similarly, let $x_i(p_X, p_Y, w, r)$ denote the demand for good *x* by individual *i*. Then the aggregate demand for good *x* is

$$D_X(p_X, p_Y, w, r) = \sum_{i=1}^{N} x_i(p_X, p_Y, w, r)$$

Finally let $l_i(p_X, p_Y, w, r)$ denote the demand for leisure by individual *i*. Then the aggregate *supply of labour* is

$$S_L(p_X, p_Y, w, r) = \sum_{i=1}^{N} \overline{L}_i - \sum_{i=1}^{N} l_i(p_X, p_Y, w, r)$$

<u>Example</u>

Suppose all individuals have Cobb-Douglas preferences. Then from (3.81) above we know that the demand for *y* by individual *i* is

(3.85) $$y_i(p_X, p_Y, w, r) = \frac{\beta_i M_i}{(\alpha_i + \beta_i + \delta_i) p_Y}$$

where we have now allowed for idiosyncratic preference-parameter values.

Now use the definition of wealth to replace $M_i$ in this expression:

(3.86) $$y_i(p_X, p_Y, w, r) = \frac{\beta_i (r\overline{K}_i + w\overline{L}_i + d_i)}{(\alpha_i + \beta_i + \delta_i) p_Y}$$

Aggregating across all individuals yields the aggregate demand for good *y*:

(3.87) $$D_Y(p_X, p_Y, w, r) = \frac{1}{p_Y}\sum_{i=1}^{N}\left(\frac{\beta_i(r\overline{K}_i + w\overline{L}_i + d_i)}{\alpha_i + \beta_i + \delta_i}\right)$$

Note that $p_Y$ is the only parameter that can be taken outside the summation operator; we cannot simplify this expression any further. In particular, we cannot express the aggregate demand in terms of aggregate wealth; the *distribution of wealth* – who owns what at the endowment – matters for the determination of aggregate demand. This in turn means that the distribution of wealth matters for the determination of equilibrium prices.

If we impose an additional restrictive assumption on preferences – that preference parameters are identical across individuals – then expression (3.87) *can* be simplified further, allowing aggregate demand to be expressed in terms of aggregate wealth:

(3.88) $$D_Y(p_X, p_Y, w, r) = \frac{1}{p_Y}\left(\frac{\beta}{\alpha + \beta + \delta}\right)V(p_X, p_Y, w, r)$$

where

(3.89) $$V(p_X, p_Y, w, r) = r\sum_{i=1}^{N}\overline{K}_i + w\sum_{i=1}^{N}\overline{L}_i + \sum_{i=1}^{N}d_i(p_X, p_Y w, r)$$

is aggregate wealth, and where

(3.90) $$\sum_{i=1}^{N}d_i(p_X, p_Y, w, r) = \Pi_X(p_X, w, r) + \Pi_Y(p_Y, w, r)$$

is the sum of aggregate profits in sectors X and Y.

Aggregate demand for *x* is constructed in the same way as for good *y*, starting from the individual-level demand from (3.82) above and aggregating across individuals to yield

(3.91) $$D_X(p_X, p_Y, w, r) = \frac{1}{p_X}\sum_{i=1}^{N}\left(\frac{\alpha_i(r\overline{K}_i + w\overline{L}_i + d_i)}{\alpha_i + \beta_i + \delta_i}\right)$$

Again, we cannot simplify this expression any further unless we assume that preference parameters are identical across individuals, in which case (3.91) reduces to

(3.92) $$D_X(p_X, p_Y, w, r) = \frac{1}{p_X}\left(\frac{\alpha}{\alpha+\beta+\delta}\right)V(p_X, p_Y, w, r)$$

where $V(p_X, p_Y, w, r)$ is aggregate wealth, given by (3.89) above.

Finally, we can construct the aggregate supply of labour, starting from the individual-level supply from (3.84) above and aggregating across individuals to yield

(3.94) $$S_L(p_X, p_Y, w, r) = \sum_{i=1}^{N} \overline{L}_i - \frac{1}{w}\sum_{i=1}^{N}\left(\frac{\delta_i(r\overline{K}_i + w\overline{L}_i + \pi_i)}{\alpha_i + \beta_i + \delta_i}\right)$$

Once again, we cannot simplify this expression any further unless we assume that preference parameters are identical across individuals, in which case (3.94) reduces to

(3.95) $$S_L(p_X, p_Y, w, r) = \sum_{i=1}^{N} \overline{L}_i - \frac{1}{w}\left(\frac{\delta}{\alpha+\beta+\delta}\right)V(p_X, p_Y, w, r)$$

where $V(p_X, p_Y, w, r)$ is aggregate wealth, given by (3.89) above.

### 3.3-4 CONSUMPTION: A NUMERICAL EXAMPLE

Suppose preferences are identical across individuals, and that those preferences are represented by

$$u(x, y, l) = x^3 y 7 l^{15}$$

That is, for this example: $\alpha = 3$, $\beta = 7$, and $\delta = 15$ for all individuals.

Recall the MRS expressions for Cobb-Douglas preferences:

$$MRS_{xy} = \frac{\alpha y}{\beta x} = \frac{3y}{7x}$$

and

$$MRS_{ly} = \frac{\delta y}{\beta l} = \frac{15y}{7l}$$

 

The he utility-maximization conditions for a individual *i* are

(3.96) $$\frac{3y}{7x} = \frac{p_X}{p_Y}$$

(3.97) $$\frac{15y}{7l} = \frac{w}{p_Y}$$

(3.99) $$p_X x + p_Y y + wl = M_i$$

These equations can be solved by substitution. Use (3.96) and (3.97) to express *x* and *l* in terms of *y*, and then substitute these into (3.99) and solve for *y*:

(3.100) $$y_i(p_X, p_Y, w, r) = \frac{7M_i}{25p_Y}$$

Now substitute this solution for *y* in (3.96) and solve for *x*, and in (3.97) and solve for *l*:

(3.101) $$x_i(p_X, p_Y, w, r) = \frac{3M_i}{25p_X}$$

(3.102) $$l_i(p_X, p_Y, w, r) = \frac{3M_i}{5w}$$

Now use (3.102) to construct the labour supply function for this individual:

(3.103) $$L_i(p_X, p_Y, w, r) = \overline{L}_i - l_i(p_X, p_Y, w) = \overline{L}_i - \frac{3M_i}{5w}$$

Because we have assumed identical preferences across individuals, the aggregate demands for Y and X can be expressed as functions of aggregate wealth:

(3.104) $$D_Y(p_X, p_Y, w, r) = \sum_{i=1}^{N} \frac{7M_i}{25p_Y} = \frac{7}{25p_Y}\sum_{i=1}^{N} M_i = \frac{7V(p_X, p_Y, w, r)}{25p_Y}$$

(3.105) $$D_X(p_X, p_Y, w, r) = \sum_{i=1}^{N} \frac{3M_i}{25p_X} = \frac{3}{25p_X}\sum_{i=1}^{N} M_i = \frac{3V(p_X, p_Y, w, r)}{25p_X}$$

Similarly, aggregate labour supply can be expressed as a function of aggregate wealth and the aggregate labour endowment:

$$S_L(p_X, p_Y, w, r) = \sum_{i=1}^{N} \overline{L}_i - \sum_{i=1}^{N} \frac{3M_i}{5w} = \sum_{i=1}^{N} \overline{L}_i - \frac{3V(p_X, p_Y, w, r)}{5w} \quad (3.106)$$

These demand and supply functions look deceptively simple but recall from (3.89) above that aggregate wealth includes the flow of aggregate profits. In particular,

$$V(p_X, p_Y, w, r) = r\sum_{i=1}^{N} \overline{K}_i + w\sum_{i=1}^{N} \overline{L}_i + \Pi_X(p_X, w, r) + \Pi_Y(p_Y, w, r) \quad (3.107)$$

In the case of the production example from Section 3.3-2 above, these profit values are

$$\Pi_Y(p_Y, w, r) = n_Y\left(\frac{p_Y^2}{8(wr)^{\frac{1}{2}}} - wF_Y\right) \quad (3.108)$$

and

$$\Pi_X(p_X, w, r) = n_X\left(\frac{p_X^4}{64w^2r} - wF_X\right) \quad (3.109)$$

If we substitute these values for profit into the aggregate wealth term in (3.104) and (3.105), the resulting expressions can be highly non-linear. For example, if we plot $D_Y(p_X, p_Y, w, r)$ against $p_Y$ it will not have the standard negative slope that we typically associate with demand curves in partial equilibrium; it could instead be U-shaped. At high enough values of $pY$, demand for $y$ can actually start to rise because the rising profits from that industry flowing to individuals boosts their demand for all goods, including good *y*.

### 3.3-5 EQUILIBRIUM

The equilibrium is characterized by a set of prices and associated quantities such that:

- the aggregate demand for good *y* is equal to the aggregate supply of good *y*
- the aggregate demand for good *x* is equal to the aggregate supply of good *x*
- the aggregate demand for capital is equal to the aggregate supply of capital
- the aggregate demand for labour is equal to the aggregate supply of labour

If the number of firms in each sector is for some reason fixed, then the equilibrium prices and quantities will be a function of those firm numbers. If instead there is free entry into both sectors, thereby driving profits to zero, then the number of firms will be determined as part of the equilibrium.

We will focus on a **free-entry equilibrium**, and so in addition to the four equilibrium conditions listed above, we also have two zero-profit conditions:

- all firms in sector Y earn zero profit
- all firms in sector X earn zero profit

Note that the zero-profit conditions mean that the profit terms that enter wealth will vanish, and the expression for aggregate wealth will reduce to

$$V(w,r) = r\sum_{i=1}^{N} \overline{K}_i + w\sum_{i=1}^{N} \overline{L}_i = r\tilde{K} + w\tilde{L}$$

This will make it much easier to solve for the equilibrium.

The algebra needed to find the equilibrium can still be somewhat over-whelming even when profits are zero, and even when we assume Cobb-Douglas technologies and Cobb-Douglas preferences. So we will work through a numerical example only, using the same parameter values that we have used in the production and consumptions examples above. This will make it easier to focus on the *method* we need to find the equilibrium.

 

Our starting point is to recognize that equilibrium prices are determined only in relative terms. In particular, recall from the simple exchange economy that we were only able to find an equilibrium price ratio $p_X / p_Y$. In that setting we made good *y* the numeraire by setting $p_Y = 1$, and then we found the equilibrium value of $p_X$.

In the two-sector model it makes most sense to specify labour as the numeraire by setting $w = 1$, and then find the equilibrium values of $p_Y$, $p_X$ and *r*.

We proceed by following 18 steps:

1. Set profit in sector Y equal to zero, and solve for $p_Y(r)$. This price will be a function of *r* (and also a function *w* but we will set $w = 1$).
   See Figure 3-27.
2. Set profit in sector X equal to zero, and solve for $p_X(r)$. This price will be a function of *r* (and also a function of *w* but again we will set $w = 1$).
   See Figure 3-28.
3. Substitute $p_Y = p_Y(r)$ into the aggregate supply of *Y* so that this aggregate supply is now expressed as a function of *r* and $n_Y$, denoted $S_Y(r, n_Y)$.
   See Figure 3-29.
4. Substitute $p_Y = p_Y(r)$ and $p_X = p_X(r)$ into the aggregate demand for *Y* so that this aggregate demand is now expressed as a function of *r*, denoted $D_Y(r)$.
   See Figure 3-29.
5. Set $S_Y(r, n_Y) = D_Y(r)$ and solve for $n_Y$ as a function *r*, denoted $n_Y(r)$.
   See Figure 3-29.
6. Substitute $p_Y = p_Y(r)$ and $n_Y = n_Y(r)$ into the aggregate demand for capital in sector Y so that this aggregate demand is now expressed as a function of *r*, denoted $D_K^Y(r)$.
   Figure 3-30.

 

7. Substitute $p_X = p_X(r)$ into the aggregate supply of *X* so that this aggregate supply is now expressed as a function of *r* and $n_X$, denoted $S_X(r, n_X)$.
   See Figure 3-31.
8. Substitute $p_Y = p_Y(r)$ and $p_X = p_X(r)$ into the aggregate demand for *X* so that this aggregate demand is now expressed as a function of *r*, denoted $D_X(r)$.
   See Figure 3.31.
9. Set $S_X(r, n_X) = D_X(r)$ and solve for $n_X$ as a function of *r*, denoted $n_X(r)$.
   See Figure 3-31.
10. Substitute $p_X = p_X(r)$ and $n_X = n_X(r)$ into the aggregate demand for capital in sector X so that this aggregate demand is now expressed as a function of *r*, denoted $D_K^X(r)$.
    See Figure 3-32.
11. Construct the aggregate demand for capital in the economy as the sum of the demands for capital in sector Y and sector X: $D_K(r) = D_K^X(r) + D_K^Y(r)$.
    See Figure 3-33.
12. Set $D_K(r) = \sum_{i=1}^{N} \overline{K}_i$ and solve for *r*. (That is, we set the demand for capital equal to the supply of capital). This solution for *r* is the equilibrium value of *r* in this economy, denoted $r^*$.
    See Figure 3-34.
13. Substitute $r = r^*$ into $p_Y(r)$ to find the equilibrium value of $p_Y$, denoted $p_Y^*$.
14. Substitute $r = r^*$ into $p_X(r)$ to find the equilibrium value of $p_X$, denoted $p_X^*$.
15. Substitute $r = r^*$ into $n_Y(r)$ to find the equilibrium value of $n_Y$, denoted $n_Y^*$.
16. Substitute $r = r^*$ into $n_X(r)$ to find the equilibrium value of $n_X$, denoted $n_X^*$.
17. Evaluate the aggregate supply of labour and the aggregate demand for labour at the calculated values for $\{r^*, p_Y^*, p_X^*, n_Y^*, n_X^*\}$, and verify that supply and demand are equal. (This serves as a test that our calculations have indeed found the correct equilibrium values).

18. Substitute $r = r^*$ into the aggregate demands for *y* and *x* to determine the quantity of goods consumed (and produced) in equilibrium.

This procedure for finding the equilibrium is not the only procedure that would work. However, this particular approach has the advantage of putting $r$ at the centre of things.

**3.3-6 EQUILIBRIUM: A NUMERICAL EXAMPLE**

Recall the production technology in sector Y,

$$Y = f_Y(K, L) = K^{\frac{1}{4}}L^{\frac{1}{4}}$$

and the production technology in sector X,

$$X = f_X(K, L) = K^{\frac{1}{4}}L^{\frac{1}{2}}$$

There are quasi-fixed managerial-labour input requirements of $F_Y = 8$ and $F_X = 4$ for firms in sectors Y and X respectively.

Recall that preferences are identical across the *N* individuals, and that those preferences are represented by

$$u(x, y, l) = xy^{\frac{7}{3}}l^5$$

The total amount of available capital is $\sum_{i=1}^{N} \overline{K}_i \equiv \tilde{K}$ and the total amount of available potential labour is $\sum_{i=1}^{N} \overline{L}_i \equiv \tilde{L}$ .

We will now find the free-entry competitive equilibrium for this economy.

 

<u>Step 1</u>

- Set profit in sector Y equal to zero, and solve for $p_Y(r)$.

Recall from Section 3.3-2 above that the profit function for a firm in sector Y is

$$\pi_Y(p_Y, w, r) = \frac{p_Y^2}{8(wr)^{\frac{1}{2}}} - wF_Y$$

Set $w=1$ then set $\pi_Y(p_Y, w, r) = 0$ and solve for $p_Y$ to yield

$$p_Y(r) = r^{\frac{1}{4}}(8F_Y)^{\frac{1}{2}}$$

Then make the substitution for $F_Y = 8$ to yield

$$p_Y(r) = 8r^{\frac{1}{4}}$$

Graphically, we have found the price at which $p_Y = MC_Y = AC_Y$ in Figure 3-27, where the position of these cost curves is determined by *r*. In particular, profit maximization requires $p_Y = MC_Y$, and zero-profit requires $p_Y = AC_Y$. Together, these two conditions identify $p_Y(r)$.

<u>Step 2</u>

- Set profit in sector X equal to zero, and solve for $p_X(r)$.

Recall from Section 3.3-2 above that the profit function for a firm in sector X is

$$\pi_X(p_X, w, r) = \frac{p_X^4}{64w^2 r} - wF_X$$

Set $w=1$ and then set $\pi_X(p_X, w, r) = 0$ and solve for $p_X$ to yield

$$p_X(r) = r^{\frac{1}{4}}(64F_X)^{\frac{1}{4}}$$

Then make the substitution for $F_X = 4$ to yield

$$p_X(r) = 4r^{\frac{1}{4}}$$

See Figure 3-28.

Step 3

- Substitute $p_Y = p_Y(r)$ into the aggregate supply of *Y* so that this aggregate supply is now expressed as a function of *r* and $n_Y$, denoted $S_Y(r, n_Y)$.

Recall from Section 3.3-2 above that aggregate supply in sector Y is

$$S_Y(p_Y, w, r) = n_Y\left(\frac{p_Y}{4(wr)^{\frac{1}{2}}}\right)$$

Set $w = 1$ and then substitute $p_Y = p_Y(r)$ to yield

$$S_Y(r, n_Y) = \frac{2n_Y}{r^{\frac{1}{4}}}$$

See Figure 3-29.

Step 4

- Substitute $p_Y = p_Y(r)$ and $p_X = p_X(r)$ into the aggregate demand for *Y* so that this aggregate demand is now expressed as a function of *r*, denoted $D_Y(r)$.

Recall from Section 3.3-4 above that aggregate demand in sector Y is

$$D_Y(p_X, p_Y, w, r) = \frac{7V(p_X, p_Y, w, r)}{25 p_Y}$$

where

$$V(p_X, p_Y, w, r) = r\sum_{i=1}^{N}\overline{K}_i + w\sum_{i=1}^{N}\overline{L}_i + \Pi_X(p_X, w, r) + \Pi_Y(p_Y, w, r)$$

is aggregate wealth.

In the free-entry equilibrium, profits are zero. Thus, aggregate wealth reduces to

$$V(p_X, p_Y, w, r) = r\sum_{i=1}^{N}\overline{K}_i + w\sum_{i=1}^{N}\overline{L}_i$$

Now set $w = 1$ and use $\sum_{i=1}^{N}\overline{K}_i \equiv \widetilde{K}$ and $\sum_{i=1}^{N}\overline{L}_i \equiv \widetilde{L}$ to express aggregate wealth as

$$V(r) = r\widetilde{K} + \widetilde{L}$$

Making this substitution, together with $p_Y = p_Y(r)$, into aggregate demand for Y then yields

$$D_Y(r) = \frac{7(r\widetilde{K} + \widetilde{L})}{200r^{\frac{1}{4}}}$$

See Figure 3-29.

<u>Step 5</u>

- Set $S_Y(r, n_Y) = D_Y(r)$ and solve for $n_Y$ as a function *r*, denoted $n_Y(r)$.

We know from Step 3 that

$$S_Y(r, n_Y) = \frac{2n_Y}{r^{\frac{1}{4}}}$$

Setting $S_Y(r, n_Y) = D_Y(r)$ and solving for $n_Y$ yields

$$n_Y(r) = \frac{7(\widetilde{L} + r\widetilde{K})}{400}$$

See Figure 3-29

<u>Step 6</u>

- Substitute $p_Y = p_Y(r)$ and $n_Y = n_Y(r)$ into the aggregate demand for capital in sector Y so that this aggregate demand is now expressed as a function of *r*, denoted $D_K^Y(r)$.

 

Recall from Section 3.3-3 above that the aggregate demand for capital in sector Y is

$$D_K^Y(p_Y, w, r) = n_Y\left(\frac{p_Y^2}{16w^{\frac{1}{2}}r^{\frac{3}{2}}}\right)$$

Set $w = 1$ and then substitute $p_Y = p_Y(r)$ and $n_Y = n_Y(r)$ into this aggregate demand to yield

$$D_K^Y(r) = \frac{7(\tilde{L} + r\tilde{K})}{100r}$$

See Figure 3-30.

<u>Step 7</u>

- Substitute $p_X = p_X(r)$ into the aggregate supply of *X* so that this aggregate supply is now expressed as a function of *r* and $n_X$, denoted $S_X(r, n_X)$.

Recall from Section 3.3-2 above that aggregate supply in sector X is

$$S_X(p_X, w, r) = n_X\left(\frac{p_X^3}{16w^2 r}\right)$$

Set $w = 1$ and then substitute $p_X = p_X(r)$ to yield

$$S_X(r, n_X) = \frac{4n_X}{r^{\frac{1}{4}}}$$

See Figure 3-31.

<u>Step 8</u>

- Substitute $p_Y = p_Y(r)$ and $p_X = p_X(r)$ into the aggregate demand for *X* so that this aggregate demand is now expressed as a function of *r*, denoted $D_X(r)$.

Recall from Section 3.3-4 above that aggregate demand in sector X is

$$D_X(p_X, p_Y, w, r) = \frac{3V(p_X, p_Y, w, r)}{25p_X}$$

We have already seen from Step 4 above that in free-entry equilibrium

$$V(r) = r\tilde{K} + \tilde{L}$$

Making this substitution, together with $p_X = p_X(r)$, into aggregate demand for X then yields

$$D_X(r) = \frac{3(r\tilde{K} + \tilde{L})}{100r^{\frac{1}{4}}}$$

See Figure 3-31.

<u>Step 9</u>

- Set $S_X(r, n_X) = D_X(r)$ and solve for $n_X$ as a function *r*, denoted $n_X(r)$.

We know from Step 7 that

$$S_X(r, n_X) = \frac{4n_X}{r^{\frac{1}{4}}}$$

Setting $S_X(r, n_X) = D_X(r)$ and solving for $n_X$ yields

$$n_X(r) = \frac{3(\tilde{L} + r\tilde{K})}{400}$$

See Figure 3-31.

<u>Step 10</u>

- Substitute $p_X = p_X(r)$ and $n_X = n_X(r)$ into the aggregate demand for capital in sector Y so that this aggregate demand is now expressed as a function of *r*, denoted $D_K^X(r)$.

 

Recall from Section 3.3-3 above that the aggregate demand for capital in sector X is

$$D_K^X(w,r,p_X) = n_X\left(\frac{p_X^4}{64w^2r^2}\right)$$

Set $w=1$ and then substitute $p_X = p_X(r)$ and $n_X = n_X(r)$ into this aggregate demand to yield

$$D_K^X(r) = \frac{3(\tilde{L} + r\tilde{K})}{100r}$$

See Figure 3-32.

Step 11

- Construct the aggregate demand for capital in the economy as the sum of the demands for capital in sector Y and sector X: $D_K(r) = D_K^X(r) + D_K^Y(r)$.

We know from Step 6 that the aggregate demand for capital in sector Y is

$$D_K^Y(r) = \frac{7(\tilde{L} + r\tilde{K})}{100r}$$

Thus, aggregate demand for capital in the economy is

$$D_K(r) = \frac{\tilde{L} + r\tilde{K}}{10r}$$

See Figure 3-33.

Step 12

- Set $D_K(r) = \sum_{i=1}^{N}\overline{K}_i$ and solve for *r*. (That is, we set the demand for capital equal to the supply of capital). This solution for *r* is the equilibrium value of *r* in this economy, denoted $r^*$.

Use $\sum_{i=1}^{N}\overline{K}_i \equiv \tilde{K}$ and then set $D_K(r) = \tilde{K}$. Solve for *r* to find the equilibrium rental rate:

$$r^* = \frac{\tilde{L}}{9\tilde{K}}$$

See Figure 3-34.

Note that this equilibrium rental rate is directly related to the scarcity of capital relative to labour; a lower supply of capital relative to labour leads to higher rental rate relative to the wage.

Step 13

- Substitute $r = r^*$ into $p_Y(r)$ to find the equilibrium value of $p_Y$, denoted $p_Y^*$.

We know from Step 1 that

$$p_Y(r) = 8r^{\frac{1}{4}}$$

Substitute $r = r^*$ into $p_Y(r)$ to yield

$$p_Y^* = 8\left(\frac{\tilde{L}}{9\tilde{K}}\right)^{\frac{1}{4}}$$

Step 14

- Substitute $r = r^*$ into $p_X(r)$ to find the equilibrium value of $p_X$, denoted $p_X^*$.

We know from Step 1 that

$$p_X(r) = 4r^{\frac{1}{4}}$$

Substitute $r = r^*$ into $p_Y(r)$ to yield

$$p_X^* = 4\left(\frac{\tilde{L}}{9\tilde{K}}\right)^{\frac{1}{4}}$$

There are two points to note about these equilibrium prices for *x* and *y*. First, they are both *increasing* in $\tilde{L}$. That is, more potential labour inputs in the economy makes prices higher. Why? A higher population demands more consumption and there is only so much capital available to allow for production of that consumption.

Second, good *y* is relatively more expensive than good *x* because good *y* is valued more highly than good *x* (recall that $\beta = 7$ and $\alpha = 3$), and because labour is less productive in the production of *y* than in the production of *x* (recall that $b_1 = \frac{1}{4}$ and $b_2 = \frac{1}{2}$).

<u>Step 15</u>

- Substitute $r = r^*$ into $n_Y(r)$ to find the equilibrium value of $n_Y$, denoted $n_Y^*$.

We know from Step 5 that

$$n_Y(r) = \frac{7(\tilde{L} + r\tilde{K})}{400}$$

Substitute $r = r^*$ into $n_Y(r)$ to yield

$$n_Y^* = \frac{7\tilde{L}}{360}$$

<u>Step 16</u>

- Substitute $r = r^*$ into $n_X(r)$ to find the equilibrium value of $n_X$, denoted $n_X^*$.

We know from Step 9 that

$$n_X(r) = \frac{3(\tilde{L} + r\tilde{K})}{400}$$

Substitute $r = r^*$ into $n_X(r)$ to yield

$$n_X^* = \frac{\tilde{L}}{120}$$

<u>Step 17</u>

- Evaluate the aggregate supply of labour and the aggregate demand for labour at the calculated values for $\{r^*, p_Y^*, p_X^*, n_Y^*, n_X^*\}$, and verify that supply and demand are equal.

Recall from Section 3.3-2 that aggregate for labour is

$$D_L^Y(p_Y, w, r) = n_Y\left(\frac{p_Y^2}{16w^{\frac{3}{2}}r^{\frac{1}{2}}}\right) + n_Y F_Y$$

and

$$D_L^X(p_X, w, r) = n_X\left(\frac{p_X^4}{32w^3 r}\right) + n_X F_X$$

in sectors X and Y respectively. The economy-wide aggregate demand for labour is the sum of these sector demands

$$D_L(p_Y, p_X, w, r) = D_L^Y(p_Y, w, r) + D_L^X(p_X, w, r)$$

Setting $w = 1$, $F_Y = 8$ and $F_X = 4$, and making the substitutions for $\{r^*, p_Y^*, p_X^*, n_Y^*, n_X^*\}$ yields the equilibrium aggregate demand for labour

$$D_L^* = \frac{7\tilde{L}}{30} + \frac{\tilde{L}}{10} = \frac{\tilde{L}}{3}$$

In comparison, recall from Section 3.3-4 that the aggregate supply of labour is

$$S_L(p_X, p_Y, w, r) = \sum_{i=1}^{N} \overline{L}_i - \frac{3V(p_X, p_Y, w, r)}{5w}$$

We have already seen from Step 4 above that in the free-entry equilibrium, aggregate wealth is simply equal to

$$V(r) = r\tilde{K} + \tilde{L}$$

 

Making this substitution into aggregate labour supply and setting $w = 1$ and $r = r^*$ yields the equilibrium aggregate supply of labour

$$S_L^* = \frac{7\tilde{L}}{30} + \frac{\tilde{L}}{10} = \frac{\tilde{L}}{3}$$

Thus, aggregate labour supply and aggregate labour demand are equal in equilibrium. Individuals allocate a third of their available potential labour to work (for example, 8 hours per day).

Note that the labour market *must* be in equilibrium if all the other markets are in equilibrium. This reflects Walras' Law.

<u>Step 18</u>

- Substitute $r = r^*$ into the aggregate demands for *y* and *x* to determine the quantity of goods consumed (and produced) in equilibrium.

We know from Step 4 that aggregate demand for *y* expressed as a function of *r* is

$$D_Y(r) = \frac{7(r\tilde{K} + \tilde{L})}{200r^{\frac{1}{4}}}$$

Substitute $r = r^*$ to yield

$$C_Y^* = \frac{7(9\tilde{K})^{\frac{1}{4}}\tilde{L}^{\frac{3}{4}}}{180}$$

We know from Step 8 that aggregate demand for *x* expressed as a function of *r* is

$$D_X(r) = \frac{3(r\tilde{K} + \tilde{L})}{100r^{\frac{1}{4}}}$$

Substitute $r = r^*$ to yield

$$C_X^* = \frac{(9\tilde{K}^{\frac{1}{4}})\tilde{L}^{\frac{3}{4}}}{30}$$

**Properties of the Equilibrium**

In Section 3.4 we will examine the "welfare properties" of the equilibrium (by which we mean "efficiency properties). Before we do so, it is helpful to identify some simple accounting properties of this economy.

<u>Gross Domestic Product</u>

The first relates to consumption values and gross domestic product (GDP). We cannot compare $C_Y^*$ and $C_X^*$ directly because they are measured in different units (for example, number of apples consumed and number of TVs consumed). However, we can compare the *market value* of these two consumption levels, given by

$$p_Y^* C_Y^* = 4\left(\frac{\tilde{L}}{9\tilde{K}}\right)^{\frac{1}{4}}\left(\frac{7(9\tilde{K}^{\frac{1}{4}})\tilde{L}^{\frac{3}{4}}}{180}\right) = \frac{14\tilde{L}}{45}$$

and

$$p_X^* C_X^* = 8\left(\frac{\tilde{L}}{9\tilde{K}}\right)^{\frac{1}{4}}\left(\frac{(9\tilde{K}^{\frac{1}{4}})\tilde{L}^{\frac{3}{4}}}{30}\right) = \frac{2\tilde{L}}{15}$$

Taking the ratio of these two values yields

$$\frac{p_Y^* C_Y^*}{p_X^* C_X^*} = \frac{7}{3}$$

It is not a coincidence that this ratio is exactly equal to the ratio of the preference parameters on goods *y* and *x*:

$$\frac{\beta}{\alpha} = \frac{7}{3}$$

We can also calculate the gross domestic product (GDP) for this economy as the sum of the market value of goods produced:

$$GDP = p_Y^* C_Y^* + p_X^* C_X^* = \frac{4\tilde{L}}{9}$$

Alternatively, we can always calculate GDP using the factor-income method, which adds up all the income accruing to the factors of production:

$$GDP = w^* S_L^* + r^* \tilde{K} = \frac{\tilde{L}}{3} + \left( \frac{\tilde{L}}{9\tilde{K}} \right) \tilde{K} = \frac{4\tilde{L}}{9}$$

where $w^* = 1$ but it is included here to emphasize that labour income is the *value* of labour supplied.

Both methods of calculating GDP must always give the same answer because in equilibrium all individual budget constraints must be satisfied: what individuals earn in income is what they spend on consumption. That budget constraint must also hold in the aggregate.

Note that $\tilde{K}$ does not appear in the expression for GDP. This is an artifact of the Cobb-Douglas functions we have used to describe this economy; $\tilde{K}$ would typically show up in more general models. However, the absence of $\tilde{K}$ in the GDP expression here actually helps to illustrate an important general point.

Suppose there is an exogenous increase in $\tilde{K}$ by an amount $k > 0$. Generally, we would expect this to raise the wealth of this economy, and cause GDP to rise. Yet it makes no difference to our GDP value. Why?

The exogenous increase in $\tilde{K}$ causes equilibrium prices to fall. In particular, consider the change in $r^*$. We know from Step 12 that the equilibrium rental rate is a function of $\tilde{K}$:

$$r^*(\tilde{K}) = \frac{\tilde{L}}{9\tilde{K}}$$

Thus, the change in $r^*$ from an exogenous increase in $\tilde{K}$ is

$$\Delta r^* = r^*(\tilde{K} + k) - r^*(\tilde{K}) = \frac{\tilde{L}}{9(\tilde{K} + k)} - \frac{\tilde{L}}{9\tilde{K}} = \frac{\tilde{L}}{9} \left( \frac{1}{\tilde{K} + k} - \frac{1}{\tilde{K}} \right) < 0$$

 

With the Cobb-Douglas structure, the price fall is such that the market value of capital is unchanged:

$$r^*(\tilde{K}+k)(\tilde{K}+k) - r^*(\tilde{K})\tilde{K} = \left(\frac{\tilde{L}}{9(\tilde{K}+k)}\right)(\tilde{K}+k) - \left(\frac{\tilde{L}}{9\tilde{K}}\right)\tilde{K} = 0$$

This is why it seems that GDP is unchanged by the increase in capital.

However, the measurement of a change in GDP is only meaningful if it is measured at *constant prices*, and this is precisely how changes in GDP are measured in practice. In our setting, we need to measure the change in the total value of consumption using the pre-change prices.

Recall that the equilibrium aggregate consumptions values are

$$C_Y^*(\tilde{K}) = \frac{7(9\tilde{K})^{\frac{1}{4}}\tilde{L}^{\frac{3}{4}}}{180}$$

and

$$C_X^*(\tilde{K}) = \frac{(9\tilde{K}^{\frac{1}{4}})\tilde{L}^{\frac{3}{4}}}{30}$$

After an exogenous increase in $\tilde{K}$ these become

$$C_Y^*(\tilde{K}+k) = \frac{7(9(\tilde{K}+k))^{\frac{1}{4}}\tilde{L}^{\frac{3}{4}}}{180}$$

and

$$C_X^*(\tilde{K}+k) = \frac{(9(\tilde{K}^{\frac{1}{4}}+k))\tilde{L}^{\frac{3}{4}}}{30}$$

If we measure the change in the total value of this consumption using constant prices, then we obtain the change in *real* GDP (indicated by the “R” subscript):

$$\Delta GDP_R = p_Y^*(\tilde{K})C_Y^*(\tilde{K}+k) + p_X^*(\tilde{K})C_X^*(\tilde{K}+r)$$
$$-\left( p_Y^*(\tilde{K})C_Y^*(\tilde{K}) + p_X^*(\tilde{K})C_X^*(\tilde{K}) \right)$$

$$= 4\left(\frac{\tilde{L}}{9\tilde{K}}\right)^{\frac{1}{4}}\left(\frac{7(9(\tilde{K}+k)^{\frac{1}{4}})\tilde{L}^{\frac{3}{4}}}{180}\right) + 8\left(\frac{\tilde{L}}{9\tilde{K}}\right)^{\frac{1}{4}}\left(\frac{(9(\tilde{K}^{\frac{1}{4}}+k))\tilde{L}^{\frac{3}{4}}}{30}\right)$$

$$-\left(4\left(\frac{\tilde{L}}{9\tilde{K}}\right)^{\frac{1}{4}}\left(\frac{7(9\tilde{K}^{\frac{1}{4}})\tilde{L}^{\frac{3}{4}}}{180}\right) + 8\left(\frac{\tilde{L}}{9\tilde{K}}\right)^{\frac{1}{4}}\left(\frac{(9\tilde{K}^{\frac{1}{4}})\tilde{L}^{\frac{3}{4}}}{30}\right)\right)$$

$$= \frac{4\tilde{L}}{9}\left(1+\frac{k}{\tilde{K}}\right)^{\frac{1}{4}} - \frac{4\tilde{L}}{9}$$

If we express this as a percentage change (by dividing by the pre-change GDP) then we obtain

$$\%\Delta GDP_R = \left(1+\frac{k}{\tilde{K}}\right)^{\frac{1}{4}} - 1 > 0$$

Thus, the exogenous increase in the capital stock does indeed cause real GDP to rise.

We can actually derive a more general expression for this % change in GDP in the context of this model (though it takes quite a bit of algebra to get there):

$$\%\Delta GDP_R = \frac{\beta}{\alpha+\beta}\left(1+\frac{k}{\tilde{K}}\right)^{a_1} + \frac{\alpha}{\alpha+\beta}\left(1+\frac{k}{\tilde{K}}\right)^{a_2} - 1$$

where $a_1$ and $a_2$ are the exponents on capital in the production functions for sector Y and X respectively, and where $\beta$ and $\alpha$ are the preference parameters on goods *y* and *x* respectively.

 

Thus, the way a change in the capital stock translates into a change in GDP depends on the technologies in the economy and on the preferences in the economy.

The Role Leisure

There is one more property of our economy that we want to highlight: the role of leisure. In Step 17 we calculated the aggregate labour supply as

$$S_L^* = \frac{7\tilde{L}}{30} + \frac{\tilde{L}}{10} = \frac{\tilde{L}}{3}$$

The aggregate amount of leisure taken in this economy is just the difference between $\tilde{L}$ and $S_L^*$:

$$LE^* \equiv \tilde{L} - S_L^* = \frac{2\tilde{L}}{3}$$

Two points are noteworthy here. First, this is independent of $\tilde{K}$, meaning that an exogenous increase in $\tilde{K}$ would not change the work/leisure balance; the increase in wealth due to an exogenous increase in capital is all spent on more consumption, not on more leisure.

This rigid result is an artifact of the Cobb-Douglas structure but it is not at odds with what we actually observe in real economies as they become richer.

Second, the unit value of leisure is measured by its opportunity cost, which is the wage income given up by not supplying this potential labour to the market. Thus, the aggregate value of leisure in this economy is equal to $w^* LE^*$ (where $w^* = 1$ because we have made labour the numeraire).

If we take the ratio of the value of leisure consumed to value of the other consumption goods consumed, we obtain

$$\frac{w^* LE^*}{p_Y^* C_Y^*} = \frac{15}{7}$$

in the case of good *y*, and

$$\frac{w^* LE^*}{p_X^* C_X^*} = \frac{15}{3}$$

in the case of good *x*.

It is not a coincidence that these ratios are exactly equal to the ratios of the preference parameter on leisure to those on goods *y* and *x* respectively:

$$\frac{\delta}{\beta} = \frac{15}{7}$$

and

$$\frac{\delta}{\alpha} = \frac{15}{3}$$

Recall that we saw a similar result with respect to the ratio of consumption expenditures on *y* and *x*.

 

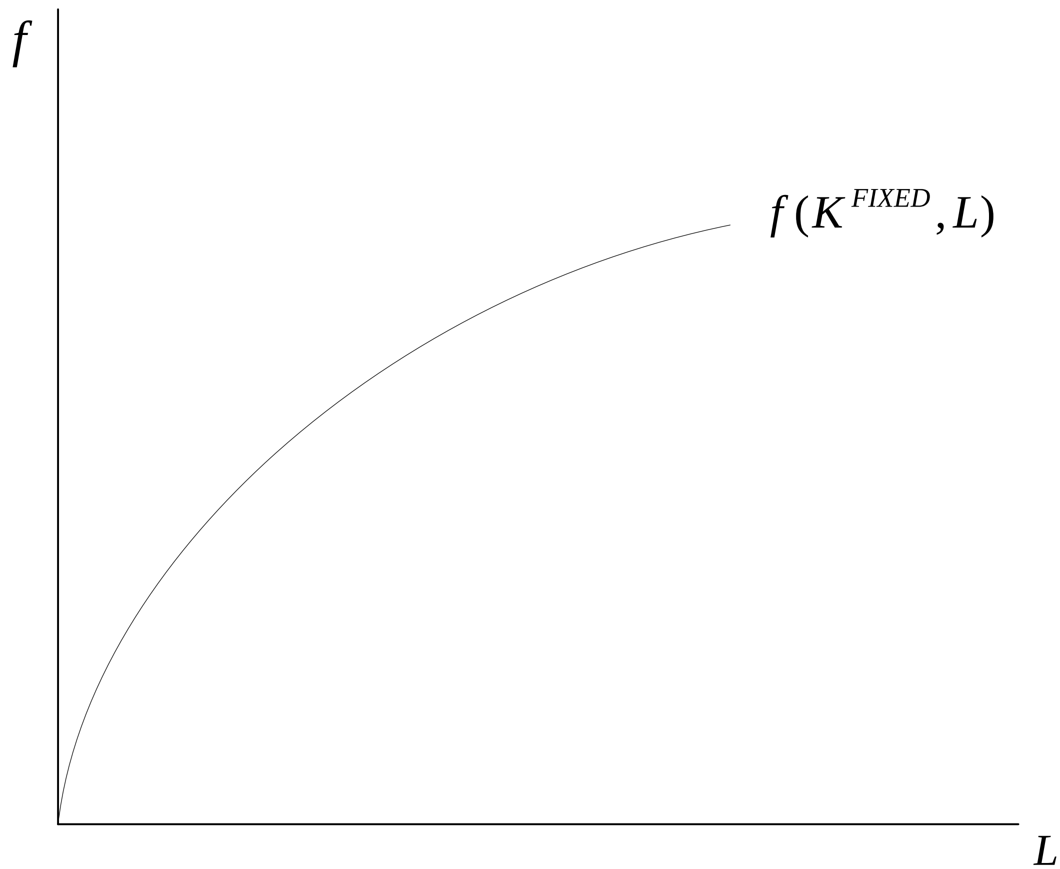


**Figure 3-9**

 

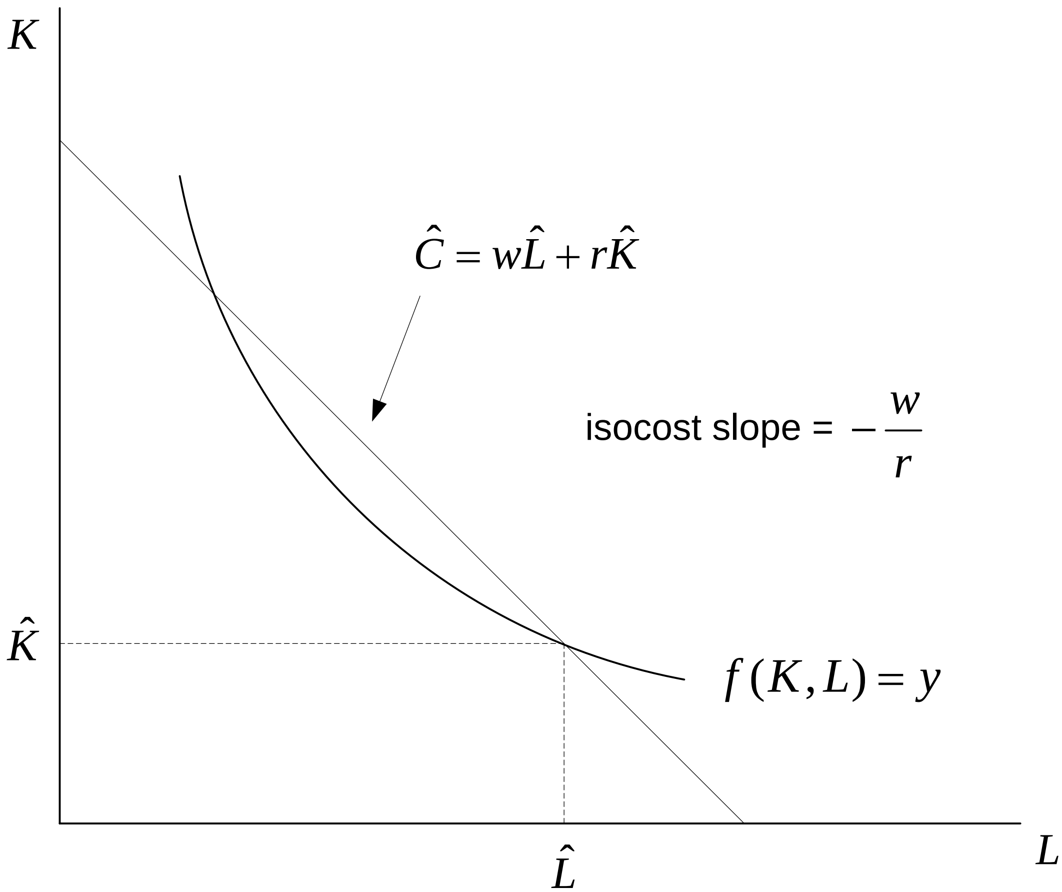


**Figure 3-10**

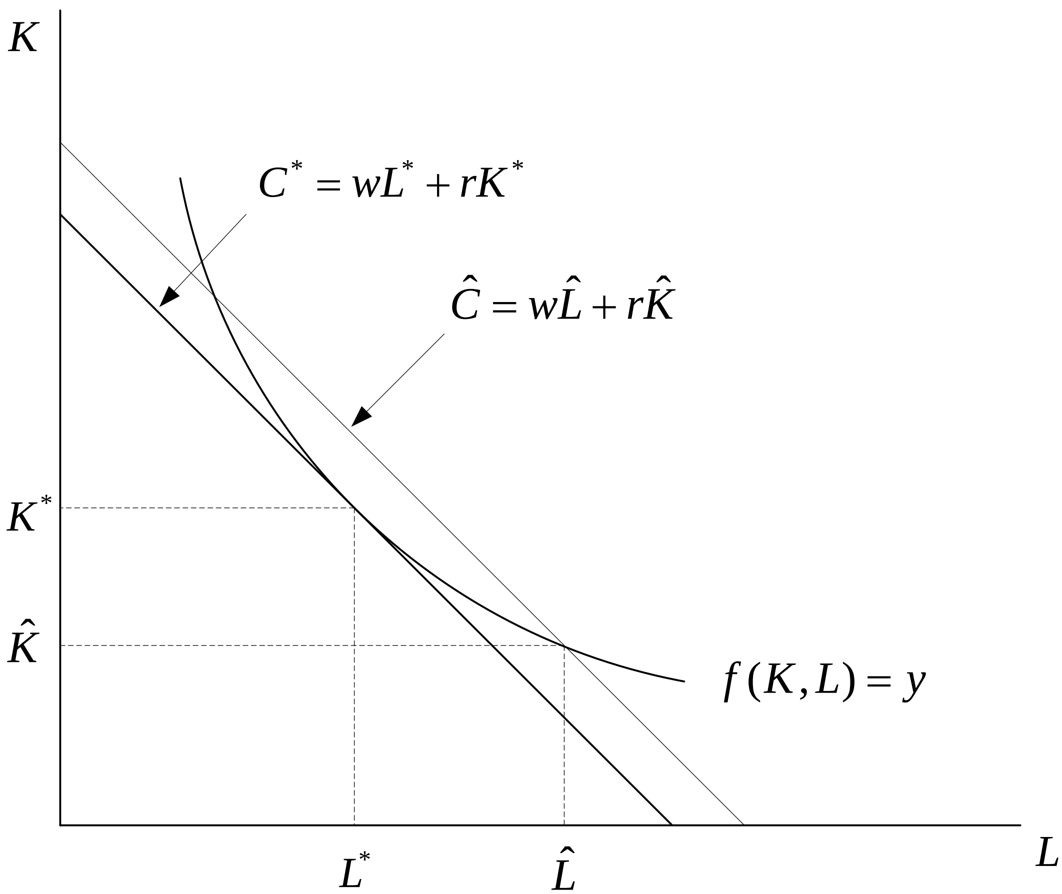


**Figure 3-11**

 

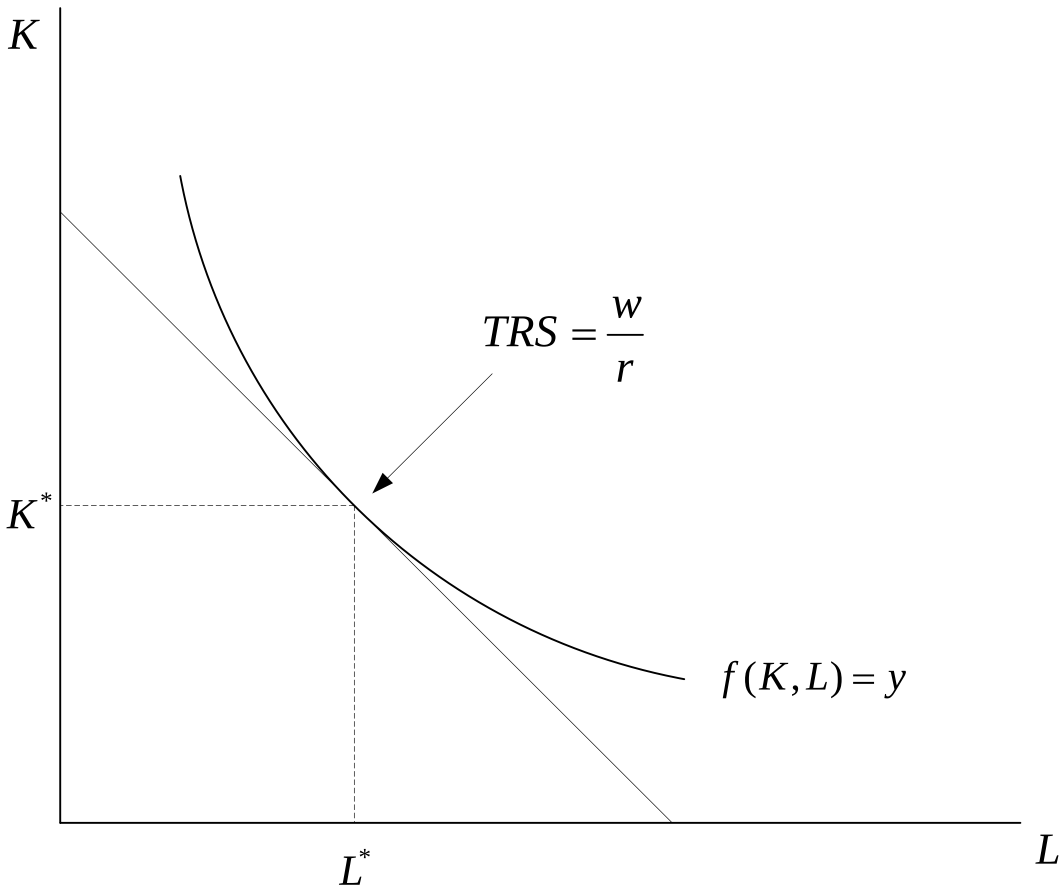


**Figure 3-12**

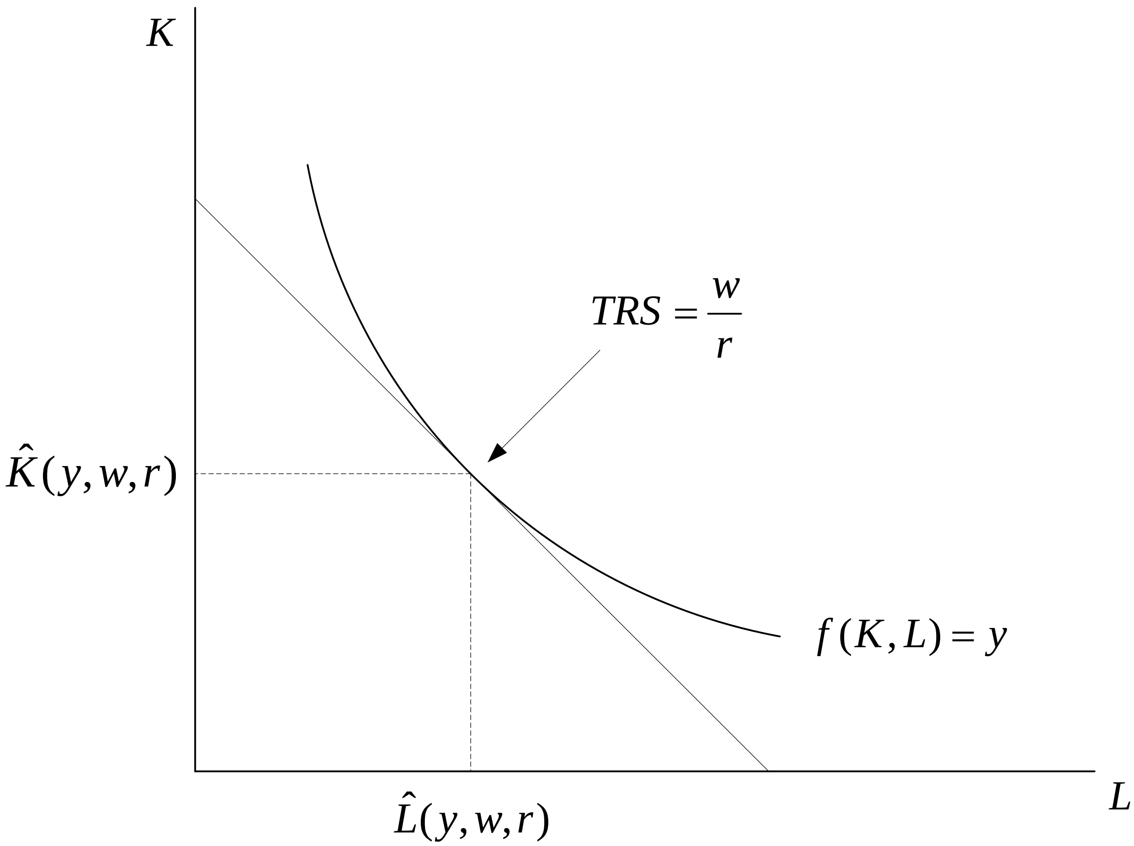


**Figure 3-13**

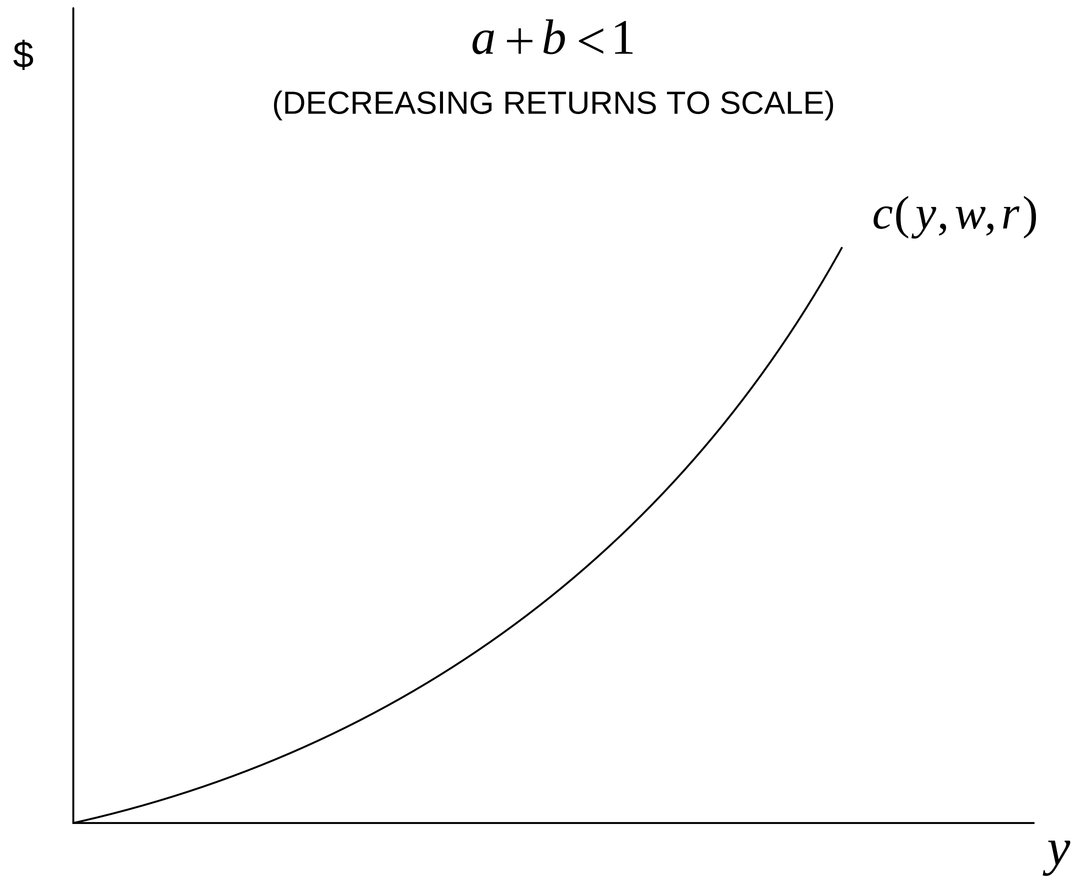


**Figure 3-14**

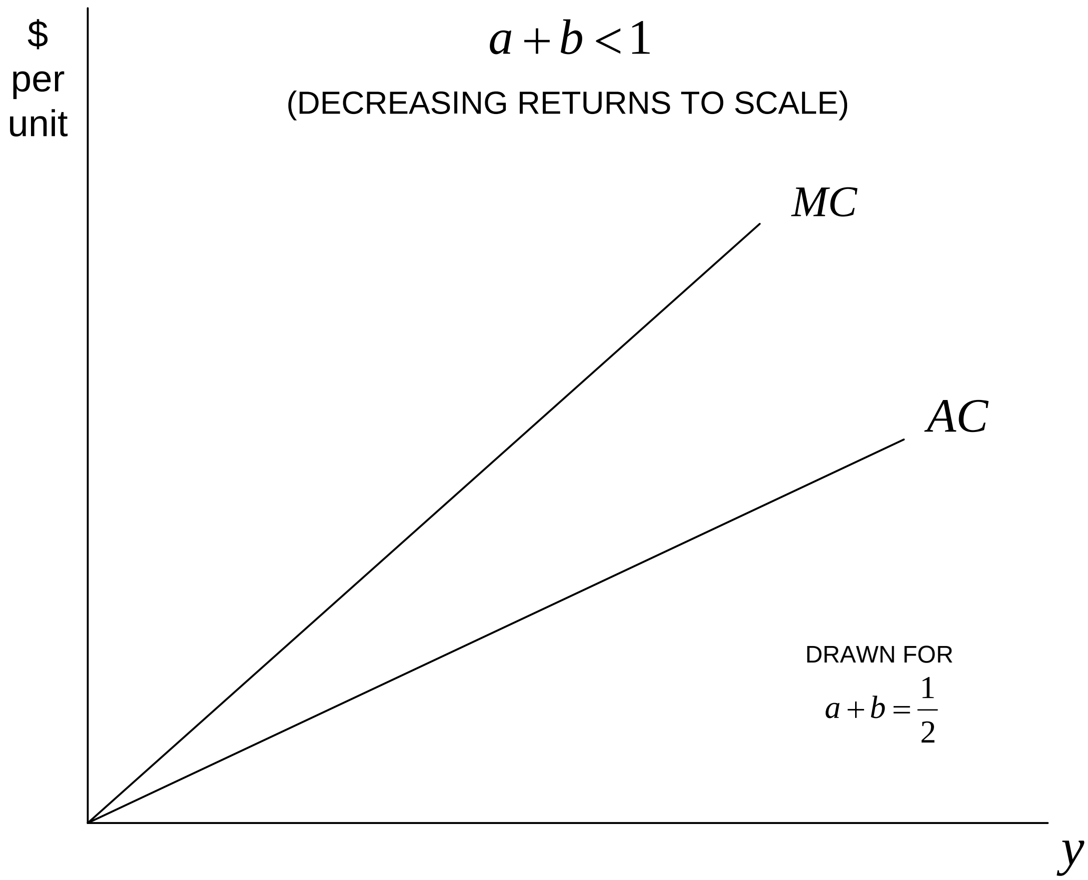
$
per
unit
$a+b<1$
(DECREASING RETURNS TO SCALE)
MC
AC
DRAWN FOR
$a+b=\frac{1}{2}$
y

**Figure 3-15**

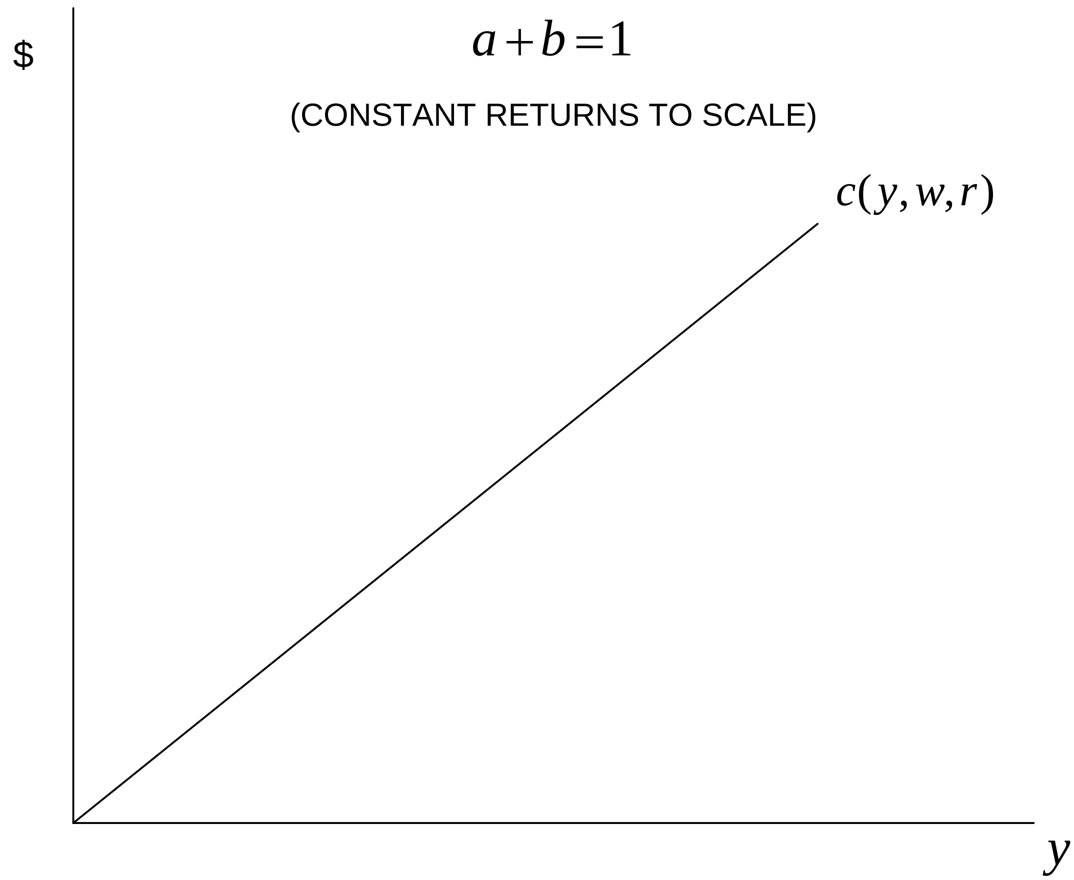


**Figure 3-16**

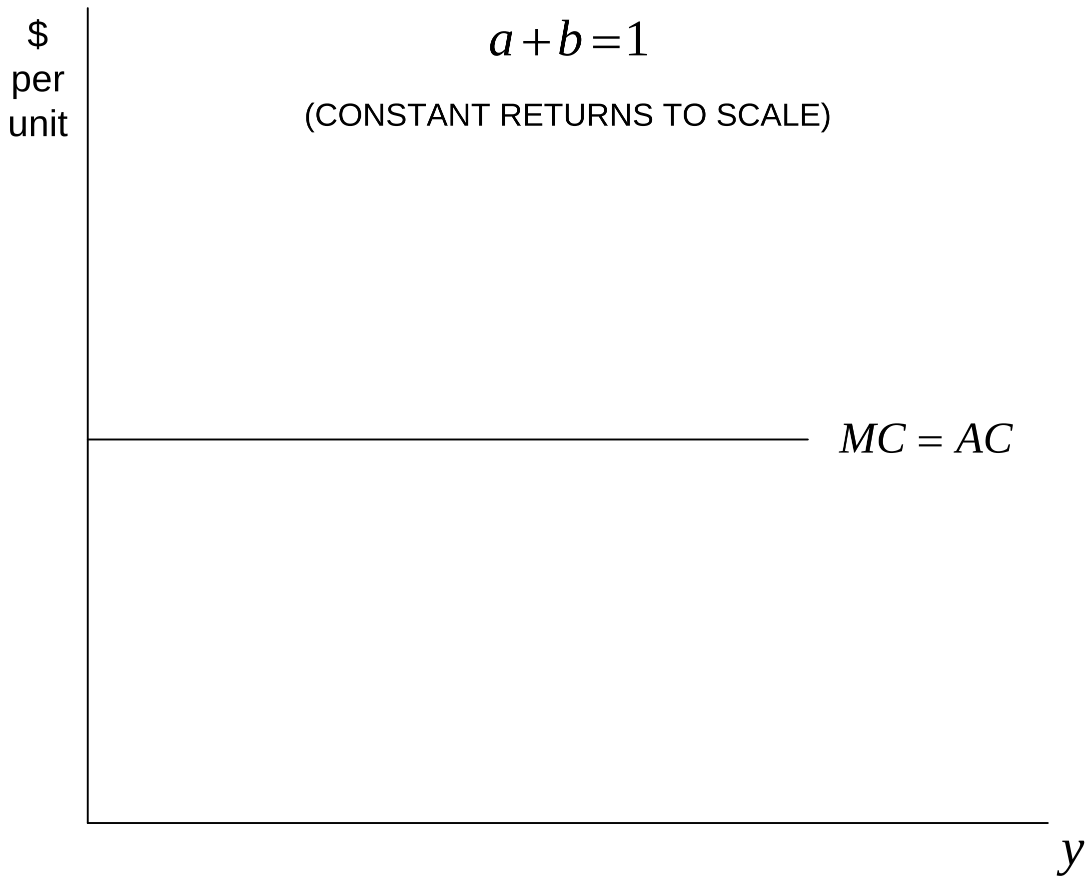


**Figure 3-17**

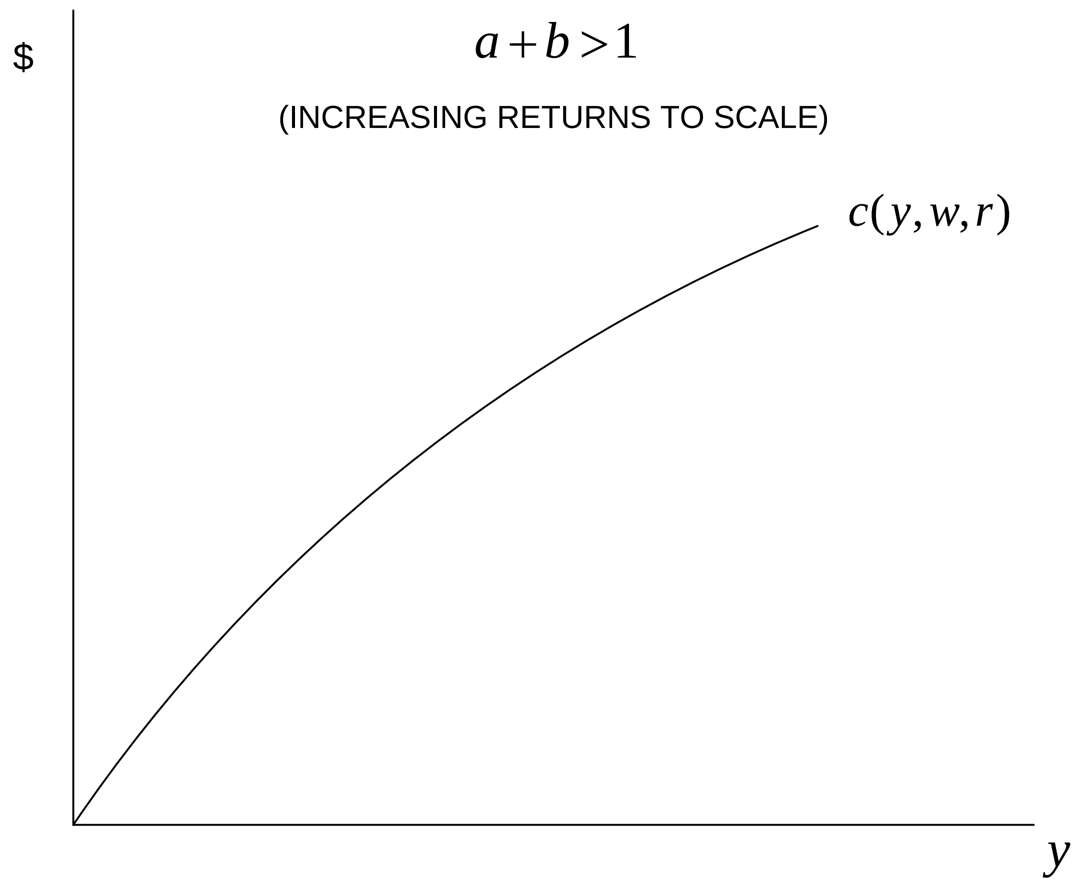


**Figure 3-18**

 

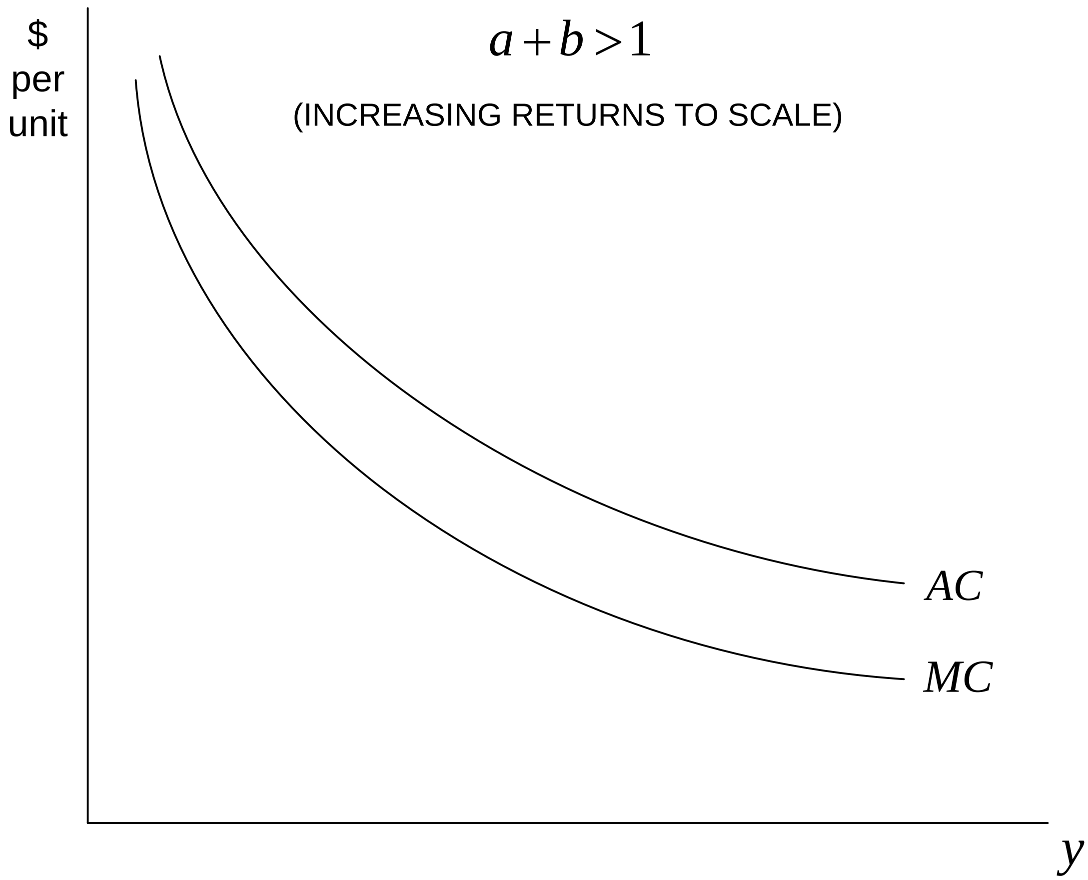


**Figure 3-19**

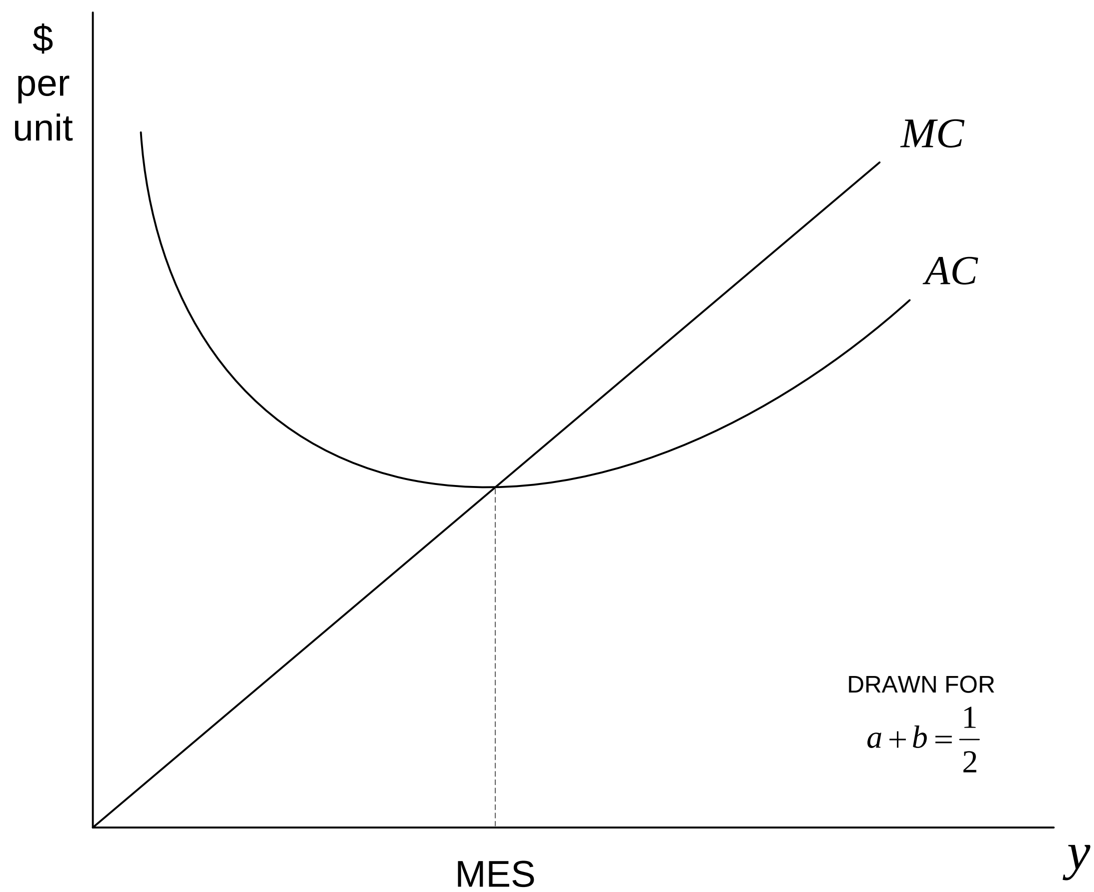


**Figure 3-20**

 

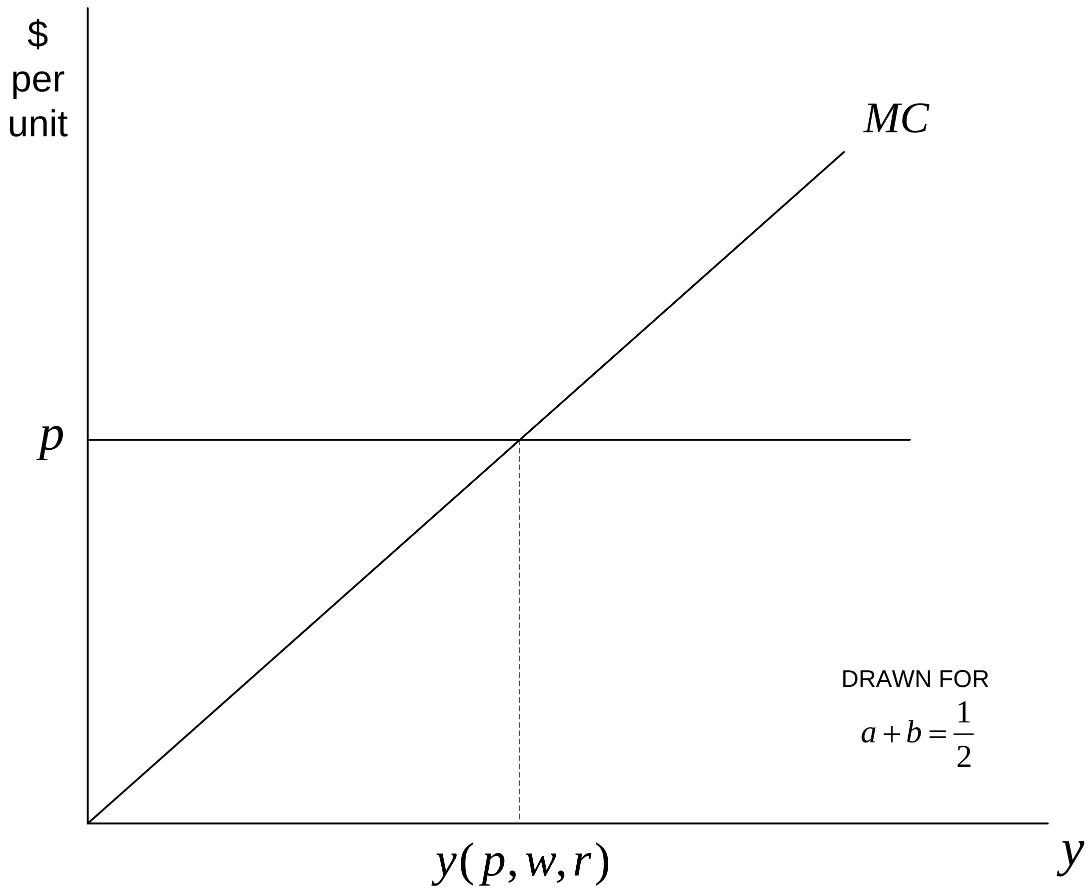


**Figure 3-21**

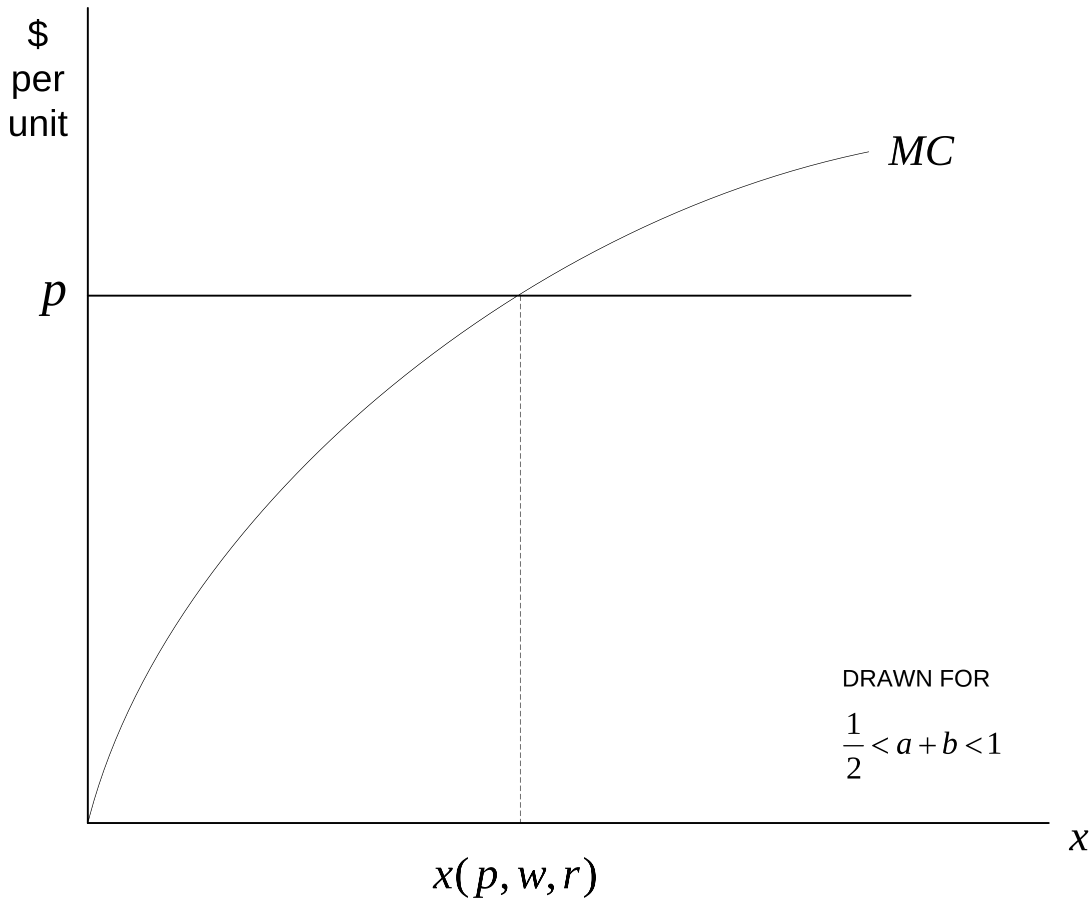


**Figure 3-22**

 

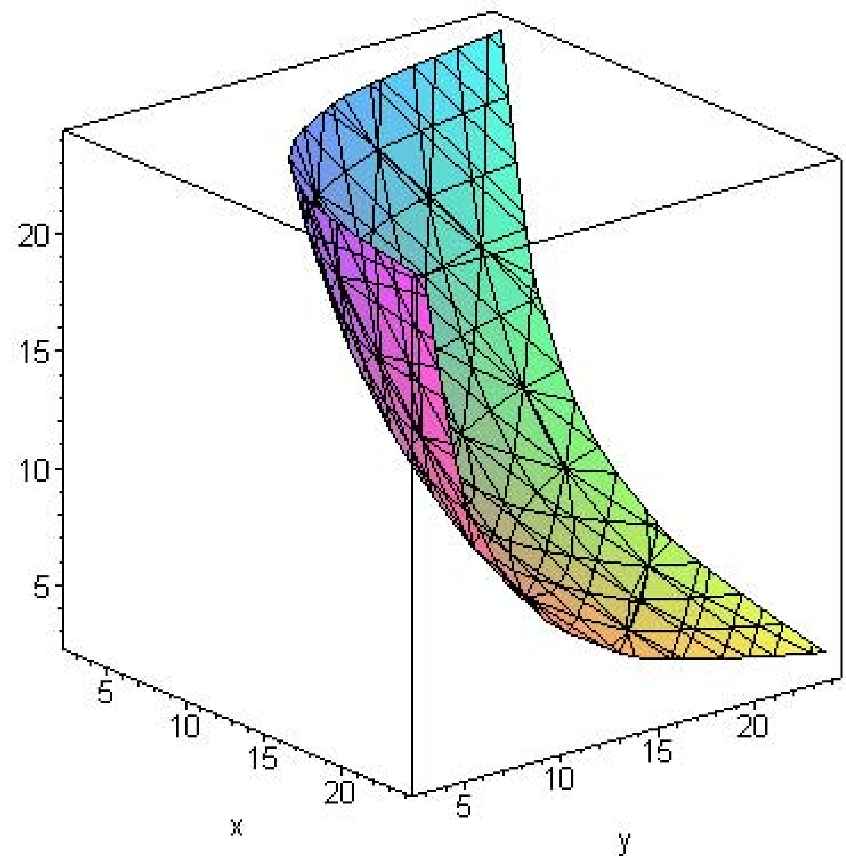


**Figure 3-23**

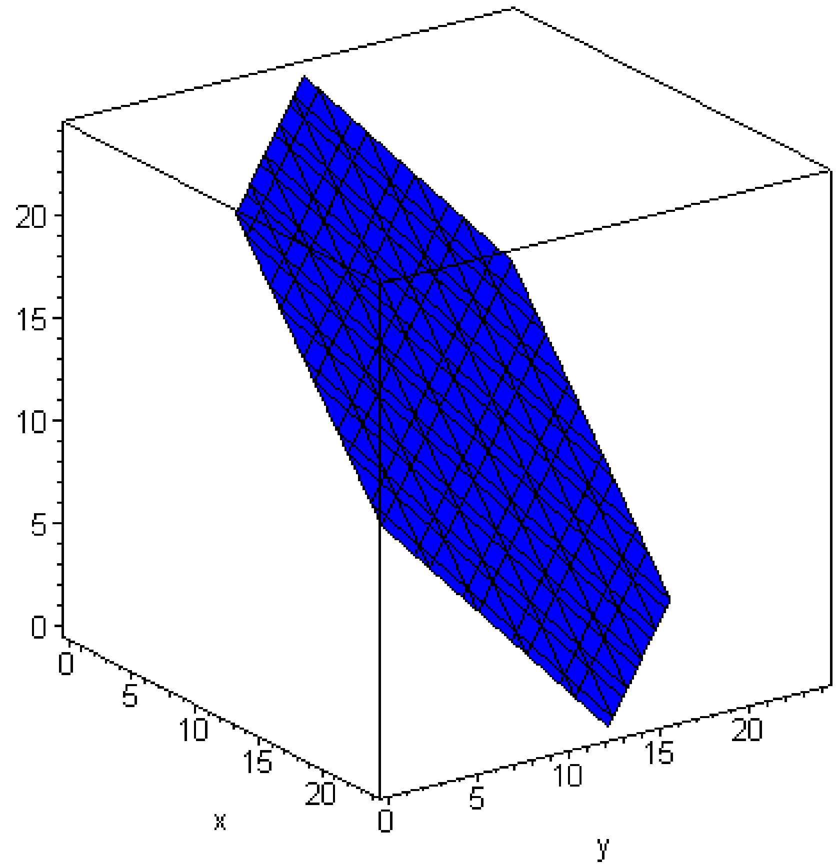


**Figure 3-24**

 

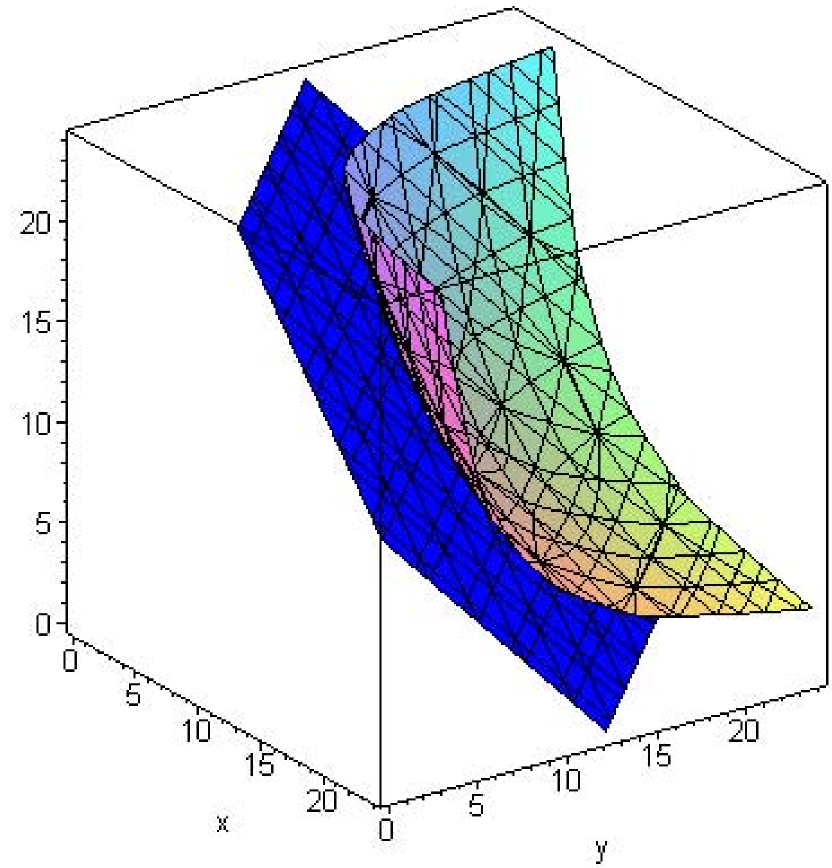


**Figure 3-25**

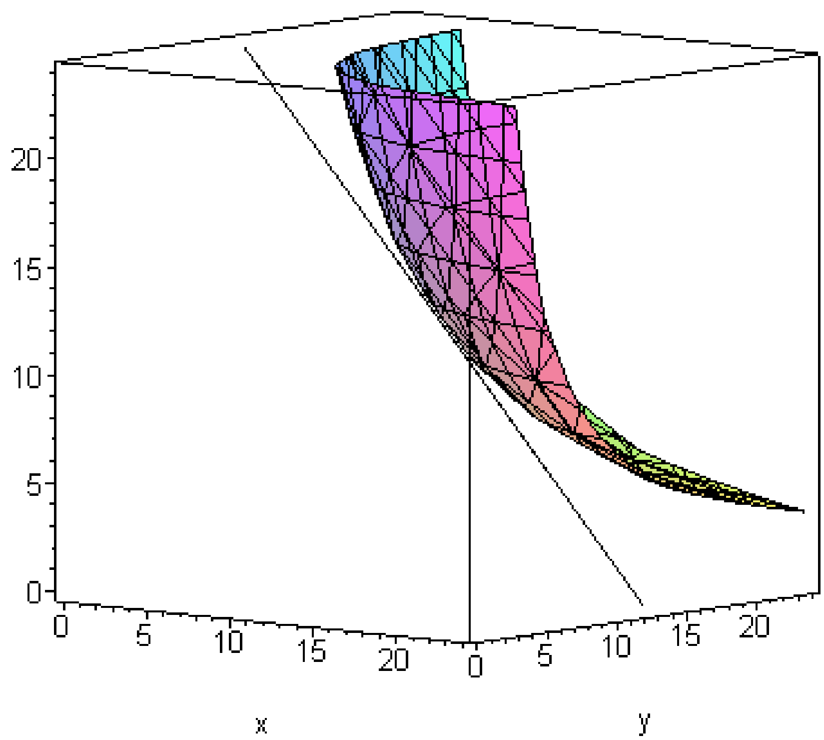


**Figure 3-26**

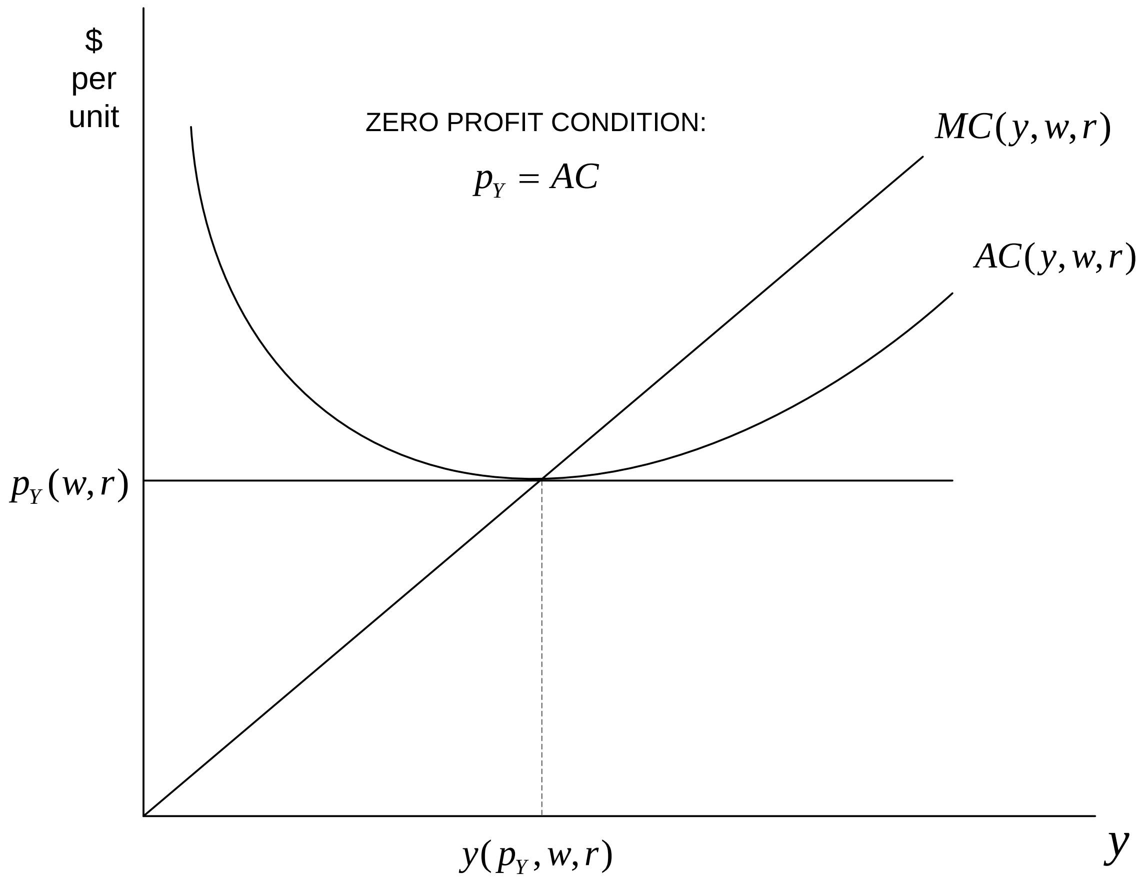
$
per
unit
ZERO PROFIT CONDITION:
$p_Y = AC$
$MC(y, w, r)$
$AC(y, w, r)$
$p_Y(w, r)$
$y(p_Y, w, r)$
$y$

**Figure 3-27**

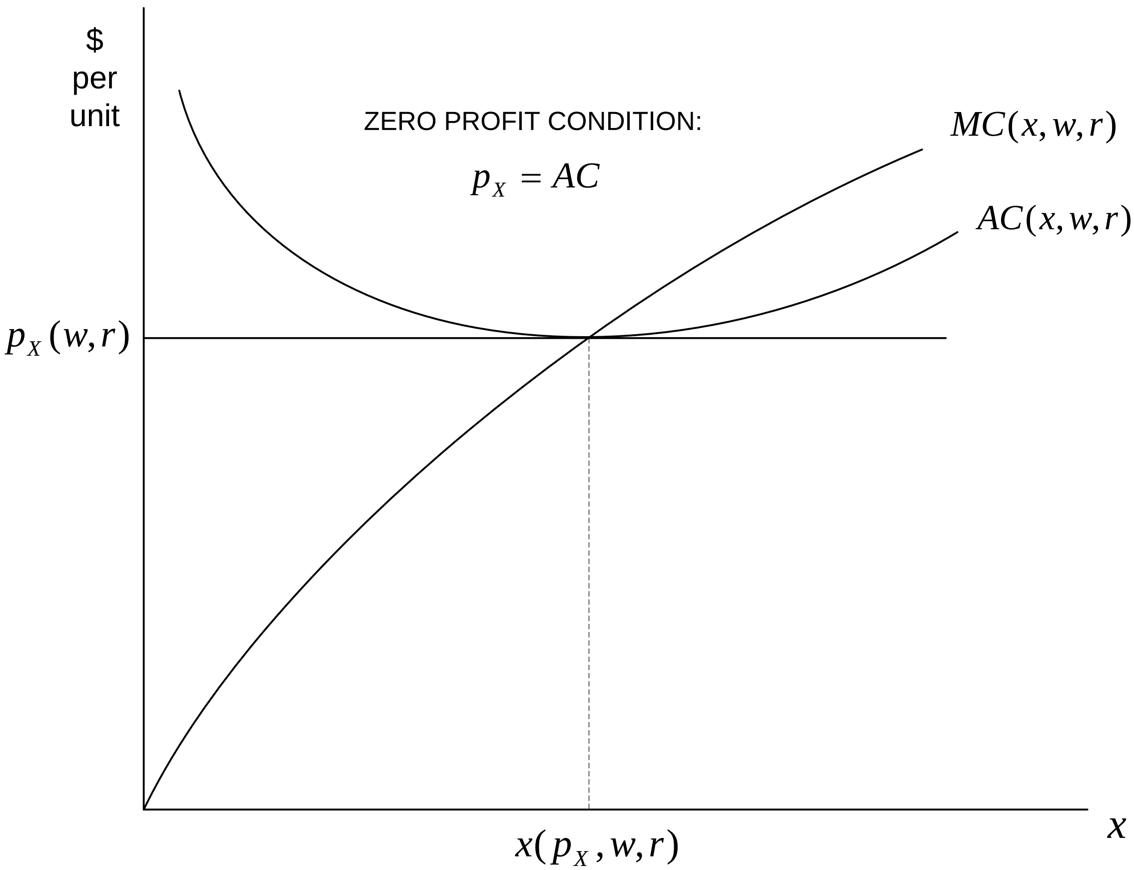
$
per
unit
ZERO PROFIT CONDITION:
$p_X = AC$
$MC(x,w,r)$
$AC(x,w,r)$
$p_X(w,r)$
$x(p_X, w, r)$
$x$

**Figure 3-28**

 

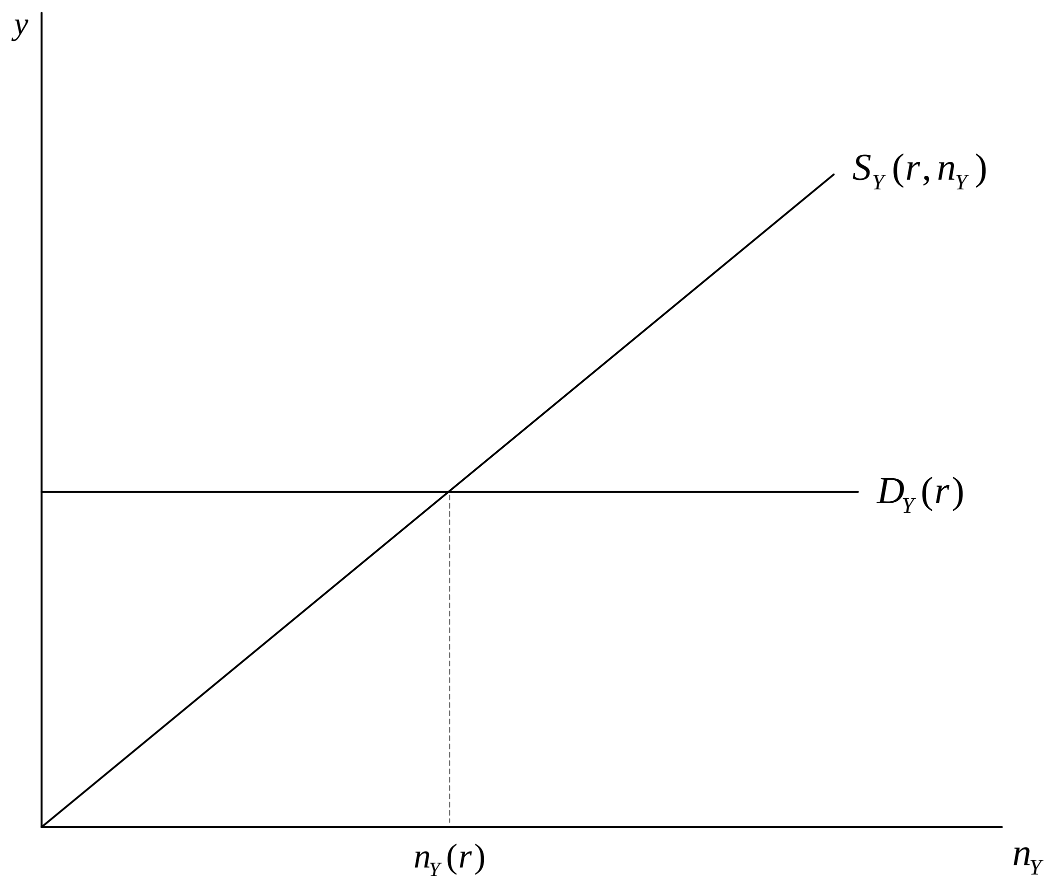


**Figure 3-29**

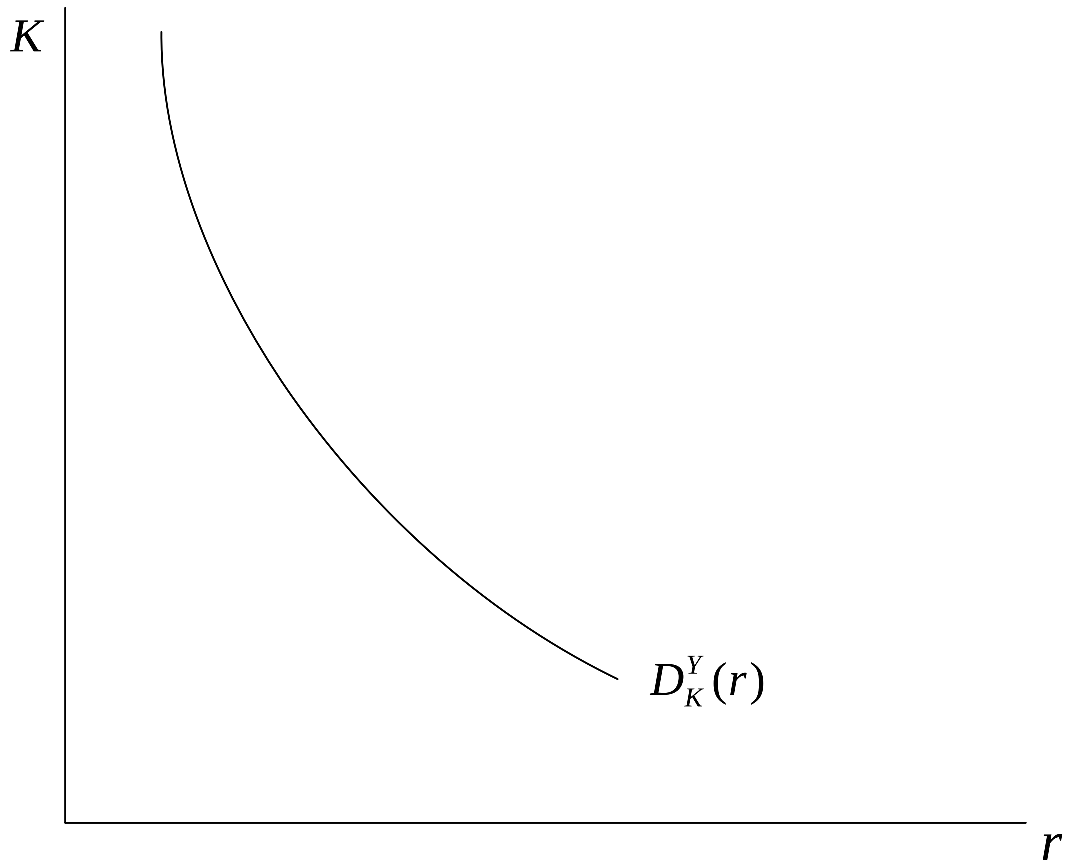


**Figure 3-30**

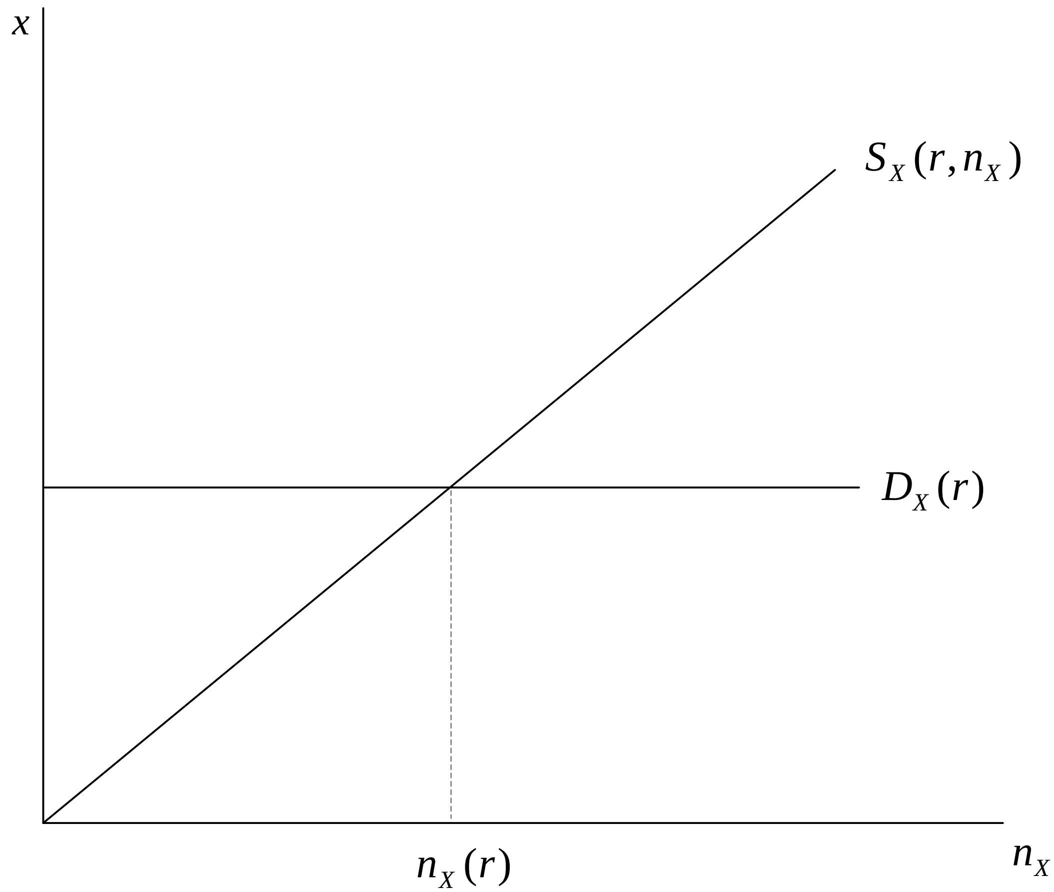


**Figure 3-31**

 

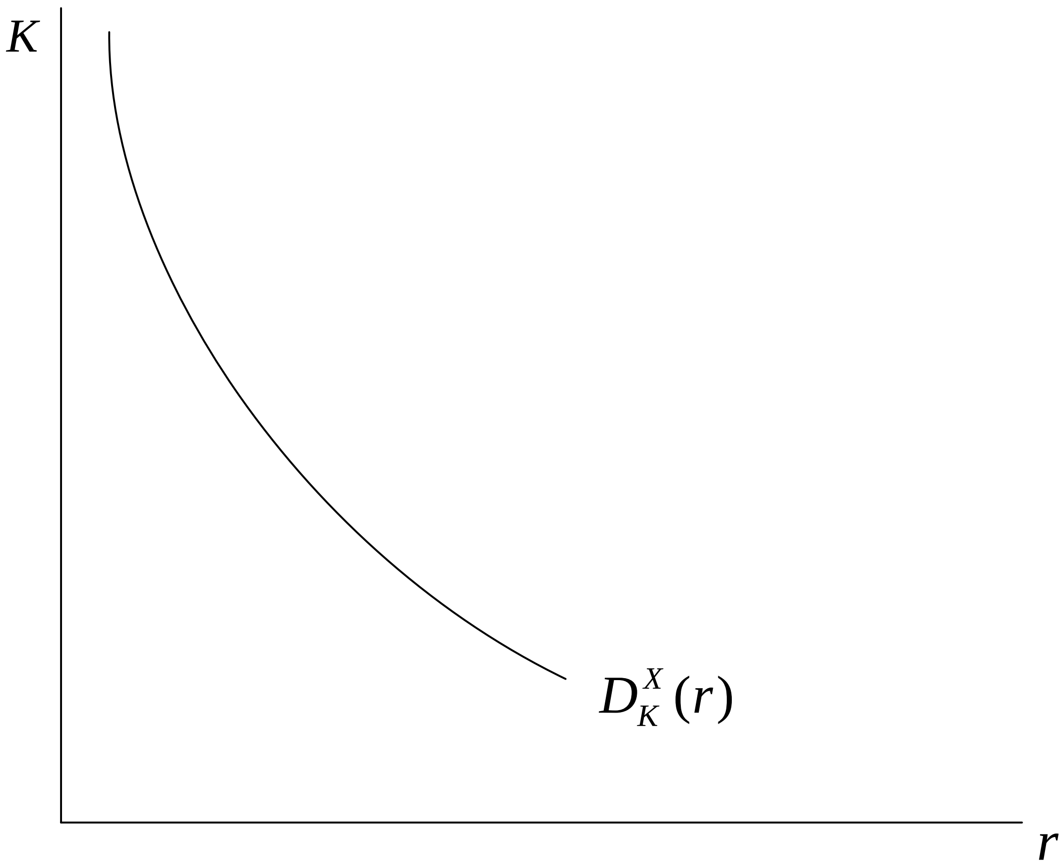


**Figure 3-32**

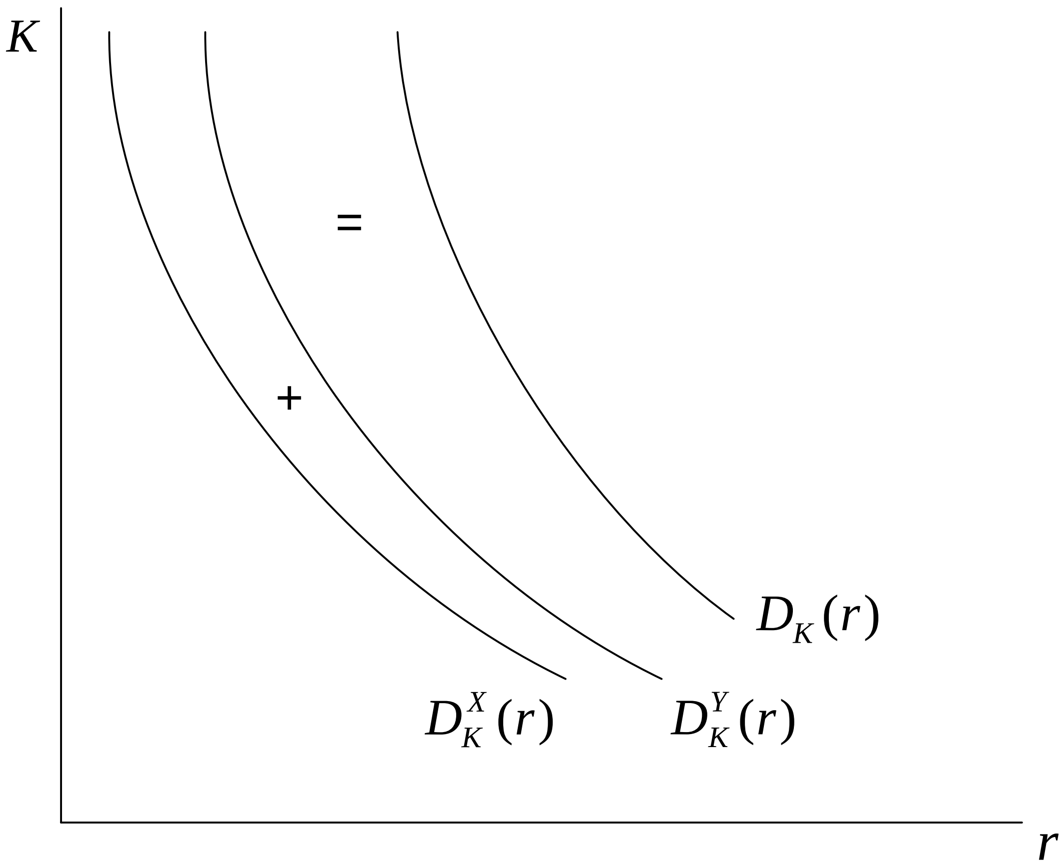


**Figure 3-33**

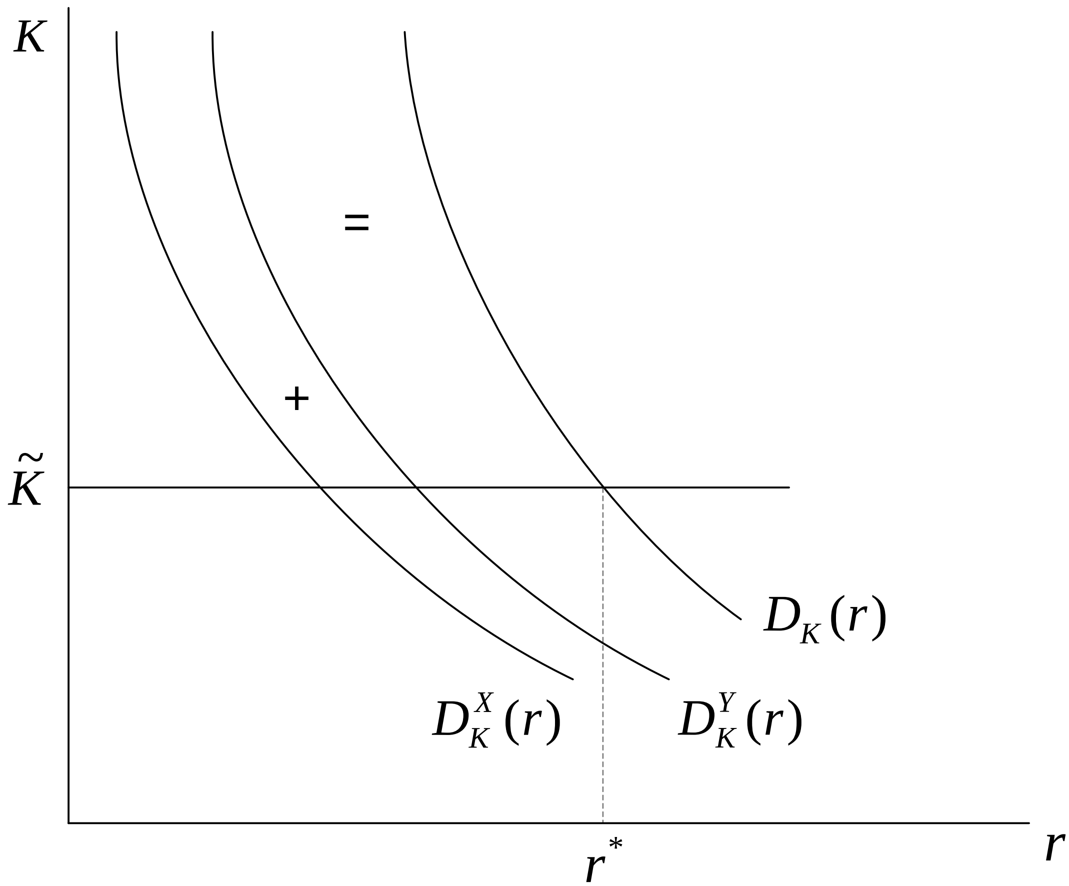


**Figure 3-34**