3.4 WELFARE PROPERTIES OF THE COMPETITIVE EQUILIBRIUM

We now want to examine the welfare properties of the competitive equilibrium. In particular, our goal is to show that the competitive equilibrium is Pareto efficient.

Pareto efficiency in this economy requires that the available factors ($\tilde{K}$ and $\tilde{L}$) are allocated in such a way that it is not possible to find an alternative allocation of those factors that makes at least one person better and no person worse off.

This requirement for efficiency is essentially no different from the requirement we imposed on our simple two-person exchange economy. In that setting we had two goods available in fixed amounts, and we had to allocate those goods across two people. Now we have two factors available in fixed amounts, and we first have to allocate those factors to production across two sectors, and then allocate the output from those two sectors across N people. Along the way, we also need to allocate some of the available potential labour to leisure.

We could in principle represent this allocation problem as one big optimization program but it turns out to be much simpler, and more instructive, to break it up into two parts and then break those two parts into sub-parts. We will proceed as follows.

Part 1: Productive Efficiency

(a) We set aside a fraction of the potential labour as leisure, and make the remaining fraction available for allocation between the two production sectors. Let $1-\lambda$ denote the fraction of potential labour set aside for leisure. Initially, λ will be unspecified.

(b) We now have factor amounts $\lambda\tilde{L}$ and $\tilde{K}$ to allocate to the production sectors, and we do so to ensure that it is not possible to produce more output in one sector without producing less in the other sector. Part of this allocation problem will involve deciding how many firms there should be in each sector (because allocating resources to a sector ultimately means allocating those resources to the firms in that sector).

(c) We use the results from Part 1(b) to construct a production possibility frontier (PPF) for the economy. This PPF will tell us the maximum aggregate quantity of good *y* the economy can produce for any given aggregate quantity of good *x* produced (conditional on the leisure allocation we made at the beginning). We will say that the economy is **productively efficient** if it is on its PPF.

Part 2: Allocative Efficiency

(a) We choose the aggregate quantities of *x* and *y* to ensure that no person can be made better off without making someone else worse off. That is, we choose a "best point on the PPF". (This "best point" will generally be dependent on how the total quantities are distributed within the population).

(b) We then return to the question of how much potential labour should be set aside for leisure (by choosing λ). This will determine the position and shape of the PPF, and we will use the results from Part 2(a) to identify the best PPF, and where on that best PPF we should be. We will say that if the economy is at a best point on the best PFF then it is **allocatively efficient**.

3.4-1 PRODUCTIVE EFFICIENCY

A production allocation is efficient if it is not possible, by re-allocating available factors, to produce more of one good without producing less of another.

In the context of our two-sector economy, this requirement can be phrased as follows: given the aggregate amounts of capital and labour used by both sectors combined, it must not be possible to increase aggregate production in sector Y without reducing aggregate production in sector X.

Our first goal is to characterize what this definition of productive efficiency means, both mathematically and graphically. We can then ask whether or not the competitive equilibrium satisfies this requirement.

Aggregate Production Functions

To begin, we need to describe aggregate production functions for each sector.

Let K_Y^+ and L_Y^+ denote the aggregate amounts of capital and labour used in sector Y. If there are n_Y identical firms each using an equal share of those total inputs, then aggregate output in sector Y is

$$Y^+ = n_Y f_Y\left(\frac{K_Y^+}{n_Y}, \frac{L_Y^+}{n_Y} - F_Y\right)$$

Note that we must take into account the quasi-fixed managerial labour requirement when calculating how much labour actually goes into production.

Similarly, let K_X^+ and L_X^+ denote the aggregate amounts of capital and labour used in sector X. If there are n_X identical firms each using an equal share of those total inputs, then aggregate output in sector X is

$$X^+ = n_X f_X\left(\frac{K_X^+}{n_X}, \frac{L_X^+}{n_X} - F_X\right)$$

In the context of our Cobb-Douglas example from Section 3.3-1, these aggregate production functions are

$$Y^+ = n_Y\left(\frac{K_Y^+}{n_Y}\right)^{a1}\left(\frac{L_Y^+}{n_Y} - F_Y\right)^{b1}$$

and

$$X^+ = n_X\left(\frac{K_X^+}{n_X}\right)^{a2}\left(\frac{L_X^+}{n_X} - F_X\right)^{b2}$$

for sectors Y and X respectively.

We can now describe **sector-level isoquants** associated with these aggregate production functions. Examples of these are depicted in **Figures 3-35** and **3-36** for sectors Y and X respectively.

We can also find the **sector-level technical rate of substitution** for each sector, which is the slope of the sector-level isoquant for that sector.

Recall from section 3.3-1 that the TRS for a firm-level production function is equal to the ratio of the marginal product of labour to the marginal product of capital. For example, the firm-level TRS in sector Y is

$$TRS^{Y} \equiv \left|\frac{dK_Y}{dL_Y}\right| = \left(\frac{\partial f_Y}{\partial L_Y} \Big/ \frac{\partial f_Y}{\partial K_Y}\right)$$

We calculate the sector-level TRS in exactly the same way, using the aggregate production function for that sector. In particular, the sector-level TRS is equal to the ratio of the marginal product of labour in the sector as a whole to the marginal product of capital in the sector as a whole.

In the case of sector Y, the sector-level TRS is

$$TRS^{Y+} = \left(\frac{\partial Y^{+}}{\partial L_Y^{+}} \Big/ \frac{\partial Y^{+}}{\partial K_Y^{+}}\right)$$

Similarly, the sector-level TRS in sector X is

$$TRS^{X+} = \left(\frac{\partial X^{+}}{\partial L_X^{+}} \Big/ \frac{\partial X^{+}}{\partial K_X^{+}}\right)$$

In the context of our Cobb-Douglas example, these sector-level TRS functions are

$$TRS^{Y+} = \frac{b_1 K_Y^{+}}{a_1 (L_Y^{+} - n_Y F_Y)}$$

and

$$TRS^{X+} = \frac{b_2 K_X^+}{a_2 (L_X^+ - n_X F_X)}$$

for sectors Y and X respectively.

An Edgeworth Box in Factor Space

Now suppose we construct an Edgeworth box, the dimensions of which measure the total amount of capital in the economy (which we denoted $\tilde{K}$) and the total amount of labour supplied in the economy.

We know that the total amount of labour supplied is less than the total amount of potential labour, $\tilde{L}$ because some is retained as leisure, so for the moment we will simply say that the total labour supplied is $\lambda\tilde{L}$, where $\lambda \in (0,1)$ is some fraction. This is the total amount of labour available for use in the two productive sectors. (We will later return to the question of how λ is determined).

Thus, the dimensions of our Edgeworth box are $\lambda\tilde{L}$ and $\tilde{K}$. See **Figure 3-37**.

Now define the SW corner of this box as the origin for sector Y. From this origin we measure the aggregate amount of labour used in that sector (on the horizontal axis) and the aggregate amount of capital used in that sector (on the vertical axis).

Similarly, define the NE corner of the box as the origin for sector X. We can then depict isoquants for each sector in the box. See **Figure 3-37**. Note that the isoquant for sector X is depicted relative to its origin, which is the NE corner of the box.

A point in this Edgeworth box represents a particular allocation of capital and labour between the two sectors (in exactly the same way we represented an allocation of goods between two people in the exchange economy).

We can now ask whether a particular allocation in the Edgeworth box is a production-efficient allocation. Consider point B in **Figure 3-38** which also depicts the isoquants for sectors Y and X that pass through that point.

It is clear from **Figure 3-38** that allocation B is not productively efficient. There are alternative allocations inside the shaded region at which aggregate output could be higher in both sectors.

This shaded region is akin to the *lens of mutual benefit* in the Edgeworth box from the exchange economy but we will not use that term here so as to avoid confusion.

However, our understanding of Pareto efficiency in the exchange economy does point to how we can identify productively efficient allocations in our Edgeworth box for factors: productively efficient allocations are those where the sector-level isoquants are tangential to each other.

There are a continuum of these tangency points, and the locus of these points constitutes the set of productively efficient allocations of the available factors, as depicted in **Figure 3-39**.

It is important to stress that this locus of productively efficient allocations is <u>not</u> a Pareto frontier (which must ultimately be constructed from preferences).

A Mathematical Description of Productive Efficiency

Now let us write down the mathematical problem whose solution yields the locus of productively efficient allocations in **Figure 3-39**:

$$\max_{K_Y^+, L_Y^+} \quad n_Y f_Y\left(\frac{K_Y^+}{n_Y}, \frac{L_Y^+}{n_Y} - F_Y\right)$$

$$\text{subject to} \quad n_X f_X\left(\frac{K_X^+}{n_X}, \frac{L_X^+}{n_X} - F_X\right) = \overline{X}$$

$$K_X^+ + K_Y^+ = \tilde{K} \quad \text{and} \quad L_X^+ + L_Y^+ = \lambda\tilde{L}$$

This optimization program states that we choose K_Y^+ and L_Y^+ (the aggregate inputs allocated to sector Y) so as to maximize aggregate output in sector Y, subject to three constraints: (1) output in sector X cannot fall below some given level $\overline{X}$; (2) the total capital used must be equal to the total available; and (3) the total labour used must be equal to the total available. We will refer to last two constraint as the "resource constraints".

In the context of the Edgeworth box, this mathematical description of the problem tells us to find the highest possible isoquant for sector Y, subject to staying on a given isoquant for sector X, and subject to staying inside the boundaries of the box. The solution is a tangency between two isoquants.

In principle, we can solve our constrained optimization problem using the Lagrange method. The solution would give us expressions for $K_Y^+(\overline{X})$ and $L_Y^+(\overline{X})$ as functions of the specified level of aggregate production, $\overline{X}$. We could then substitute these solutions back into the aggregate production function for sector Y to yield an expression that tells us the maximum possible aggregate output in sector Y for any given level of aggregate output in sector X. This relationship between aggregate output in sector X and the maximum possible output in sector Y is called the **production possibility frontier** (PPF) for this economy.

We can also derive this PPF via a less direct method, using the graphical representation from **Figure 3-39** to guide us to a solution.

A tangency between two isoquants in **Figure 3-39** is a point where the slope of one isoquant is equal to the slope of the other isoquant. We know that the slope of an isoquant is its TRS, so a tangency is a point where

(3.110) $$TRS^{Y+} = TRS^{X+}$$

If we combine this equation with the two resource constraints then we can find a description of the locus of productively efficient allocations in **Figure 3-39**. Let us do that using a numerical example.

Numerical Example

Recall that the sector-level TRS functions when production functions are Cobb-Douglas are given by

$$TRS^{Y+} = \frac{b_1 K_Y^+}{a_1(L_Y^+ - n_Y F_Y)}$$

and

$$TRS^{X+} = \frac{b_2 K_X^+}{a_2(L_X^+ - n_X F_X)}$$

for sectors Y and X respectively.

Now let us impose the parameter values from our numerical example from section 3.3-6: $\{a_1 = \tfrac{1}{4}, b_1 = \tfrac{1}{4}, a_2 = \tfrac{1}{4}, b_2 = \tfrac{1}{2}, F_Y = 8, F_X = 4\}$

The sector-level TRS functions now become

$$TRS^{Y+} = \frac{K_Y^+}{L_Y^+ - 8n_Y}$$

and

$$TRS^{X+} = \frac{2K_X^+}{L_X^+ - 4n_X}$$

We know from the resource constraints that

$$K_X^+ = \tilde{K} - K_Y^+$$

and

$$L_X^+ = \lambda\tilde{L} - L_Y^+$$

Substitute these resource constraints into TRS^{X+} to obtain

$$TRS^{X+} = \frac{2(\tilde{K} - K_Y^+)}{\lambda\tilde{L} - L_Y^+ - 4n_X}$$

Now impose the tangency condition by setting $TRS^{Y+} = TRS^{X+}$, and solve for K_Y^+:

(3.111) $$K_Y^{+PE} = \frac{2\tilde{K}(L_Y^+ - 8n_Y)}{L_Y^+ + \lambda\tilde{L} - 4n_X - 16n_Y}$$

The "PE" superscript here denotes the productively efficient allocation.

Plotting (3.111) in the Edgeworth box yields a locus of tangencies like the one depicted in **Figure 3-39**.

What about the Number of Firms?

Our description of the productively efficient allocations thus far is conditional on the number of firms in each sector. We now need to ask: what is the productively efficient number of firms?

Productive efficiency requires that it is not possible to change the number of firms within a sector in a way that allows more aggregate output to be produced in that sector using the same aggregate input values.

For example, suppose we allocate capital K_Y^+ and labour L_Y^+ to sector Y. That capital and labour then gets divided up across the n_Y firms in the sector. We can then ask: what value of n_Y will maximize aggregate output in sector Y? This is the productively efficient number of firms for that sector.

How do we find this value for n_Y? Let us again focus on the Cobb-Douglas example, where aggregate output in sector Y is

$$Y^{+} = n_Y \left(\frac{K_Y^{+}}{n_Y} \right)^{a1} \left(\frac{L_Y^{+}}{n_Y} - F_Y \right)^{b1}$$

Differentiate Y^{+} with respect to n_Y, and then set this derivative equal to zero and solve for n_Y (which takes a bit of algebra):

(3.112) $$n_Y^{PE}(L_Y^{+}) = \frac{(1-a_1-b_1)L_Y^{+}}{(1-a_1)F_Y}$$

$$= \frac{L_Y^{+}}{12} \quad \text{for the numerical example values}$$

The "PE" superscript here denotes the productively efficiency solution.

Note the key role played by F_Y here. If F_Y is very large then it makes sense to have a relatively small number of firms, since each firm requires a large quasi-fixed input before it can produce anything at all. (It makes no sense to have lots of firms, each requiring their own managers, and have no labour left over to do the actual production).
At the opposite extreme, as $F_Y \to 0$, the number of firms should be extremely large. To see why, consider the limiting case where $F_Y = 0$. In that case, the AC function in this sector is increasing right from $y = 0$, as depicted in Figure 3-15 (repeated here as **Figure 3-40**), because we have DRS (and hence increasing MC), and there is no AFC. Thus, the unit cost of production is lowest when each firm produces just a tiny amount, and so aggregate output should be spread across as many firms as possible.

In general, in a setting with DRS, the optimal number of firms strikes a balance between keeping MC low (by having many firms, each producing a small amount) and keeping AFC low (by not having *too* many firms, each with their own managers).

Now consider the productively efficient number of firms in sector X. Recall that in the Cobb-Douglas case, aggregate output in sector X is

$$X^{+} = n_X \left(\frac{K_X^{+}}{n_X} \right)^{a2} \left(\frac{L_X^{+}}{n_X} - F_X \right)^{b2}$$

Differentiate X^{+} with respect to n_X, and then set this derivative equal to zero and solve for n_X:

(3.113) $$n_X^{PE}(L_X^{+}) = \frac{(1-a_2-b_2)L_X^{+}}{(1-a_2)F_X}$$

$$= \frac{L_X^{+}}{12} \quad \text{for the numerical example values}$$

where the "PE" superscript denotes the productive efficiency solution. Again, note the important role played by the quasi-fixed input, F_X.

We can now complete our characterization of productive efficiency by substituting these optimal values for n_Y and n_X into (3.111) to obtain

(3.114) $$K_Y^{+PE} = \frac{a_!(1-a_2)\tilde{K}L_Y^{+}}{a_2(1-a_1)\lambda\tilde{L} + L_Y^{+}(a_1 - a_2)}$$

$$= \left(\frac{\tilde{K}}{\lambda\tilde{L}} \right) L_Y^{+} \quad \text{for the numerical example values}$$

Note that in the special case where $a_1 = a_2$ (as in the numerical example), the locus of tangencies in our Edgeworth box is simply a straight line along the diagonal of the box. If $a_1 < a_2$, the locus is bowed above the diagonal; if $a_1 > a_2$ then the locus is bowed below the diagonal. See **Figure 3-41**.

The Production Possibility Frontier

We can now use our equation from (3.114) to derive the production possibility frontier for this economy.

Recall that equation (3.114) describes the efficient allocation of capital to sector Y when that sector is allocated labour in the amount L_Y^+. The residual amounts of available capital and labour are allocated to sector X: $K_X^+ = \tilde{K} - K_Y^{+PE}$ and $L_X^+ = \lambda\tilde{L} - L_Y^+$.

Thus, for any allocation L_Y^+ and corresponding K_Y^{+PE}, we can work out the associated levels of output in sector Y and sector X by using the aggregate production functions for those sectors. These output levels are the productively efficient outputs associated with a given L_Y^+, and we denote them $Y^{+PE}(L_Y^+)$ and $X^{+PE}(L_Y^+)$ for sector Y and X respectively.

We can picture these output levels in our Edgeworth box, as follows. Pick any value of L_Y^+ and identify the corresponding point on the efficiency locus. The isoquants for sector Y and X at that point correspond to output values $Y^{+PE}(L_Y^+)$ and $X^{+PE}(L_Y^+)$ respectively. See **Figure 3-42**.

We can derive expressions for $Y^{+PE}(L_Y^+)$ and $X^{+PE}(L_Y^+)$ but they are complicated, and ultimately we not interested in them directly.

Our interest instead lies in the relationship between $Y^{+PE}(L_Y^+)$ and $X^{+PE}(L_Y^+)$. To find this relationship, we invert $X^{+PE}(L_Y^+)$ to express L_Y^+ in terms of X^+: $L_Y^{+PE}(X^+)$. We can then substitute this expression for L_Y^+ in $Y^{+PE}(L_Y^+)$, and this yields a relationship between Y^{+PE} and X^+, denoted $Y^{+PE}(X^+)$.

This relationship between Y^{+PE} and X^+ is the **production possibility frontier**; it tells us the maximum possible amount of aggregate production in sector Y for any given level of aggregate production in sector X, given the available amounts of capital and labour in the economy.

We can derive a fairly simple expression for the PPF in our numerical example (though the algebra needed to get there is quite a bit of work):

$$(3.115) \qquad Y^{+PE}(X^+) = \frac{(\lambda\tilde{L})^{\frac{3}{4}}(3\tilde{K})^{\frac{1}{4}}}{6} - \frac{X^+}{2}$$

This PPF is plotted in **Figure 3-43**. Note that it is linear. This linearity is a direct consequence of the fact that $a_1 = a_2$ in our numerical example. In all other cases, the PPF has a concave shape like the one depicted in **Figure 3-44**, and the mathematics needed to describe it can be quite complicated.

The slope of the PPF is called the **marginal rate of transformation** (MRT). It tells us the rate at which X^+ must be given up in order to produce more Y^+ when resources are allocated efficiently.

The terminology of the MRT reflects the idea that we can effectively transform good *x* into good *y* by taking some resources out of sector X and reallocating those resources to sector Y.

In the case where $a_1 = a_2$, the MRT is a constant. (In the numerical example, it is $-\frac{1}{2}$). In all other cases, the MRT varies along the PPF, as depicted in **Figure 3-44**.

It is useful to think of the MRT in terms of **marginal opportunity cost**. In particular, the MRT at any point on the PPF effectively measures the marginal cost of good *y* in terms of good *x*. That is, it measure how much *x* must be given up in order to produce an extra unit of *y*. This interpretation of the MRT will be quite important when we later relate the MRT to the ratio of equilibrium prices (in Section 3.4-3).

There is one final important point to note about the PPF. We have drawn the PPF in **Figure 3-43** for a given amount of available labour, $\lambda\tilde{L}$. However, we know that λ is ultimately a choice variable. A smaller value of λ (reflecting a higher fraction of potential labour retained as leisure) shifts the PPF inward, as illustrated in **Figure 3-45**.

A useful way to think about how the PPF and leisure are related is to imagine a generalized PPF plotted as a surface in 3D space, where Y^+ is plotted on the vertical axis, and X^+ and $\lambda\tilde{L}$ are plotted on the horizontal plane, as in **Figure 3-46**.

A point on this surface represents a production possibility, in terms of aggregate outputs and aggregate labour devoted to production. Ultimately, need to pick a point on this surface.

If we look at the surface from side on, we see 2-dimensional PFFs like the ones depicted in **Figure 3-45**, corresponding to increasing values of $\lambda\tilde{L}$; see **Figure 3-47**.

Thus, picking a point on the generalized PPF surface is equivalent to picking a value of λ, which determines which 2-dimensional PPF we have, and then picking a point on that 2-dimensional PPF to give us values for X^+ and Y^+.

We now turn to the question of how we pick that point.

3.4-2 ALLOCATIVE EFFICINECY

What is the efficient combination of $\lambda\tilde{L}$, X^+ and Y^+ in this economy? That is, which point on the generalized PPF should we pick?

To begin, imagine for a moment that there is just one individual in this economy, and that this individual consumes all of the goods produced. Imagine too that we keep λ fixed for the moment, and just think about choosing a point on a 2-dimensional PFF.

In this case, the efficient combination of goods is simply that which maximizes our lone person's utility. Graphically, we are looking for the highest possible indifference curve for this person, subject to being on the PPF. See **Figure 3-48** (which is drawn for the most general case, where the PPF is bowed out).

The solution to this problem involves a tangency between an indifference curve and the PPF:

$$MRS^1_{xy} = MRT_{xy}$$

The "1" superscript here indicates individual one. (We will soon introduce a second person).

This tangency condition tells us that the rate at which person 1 is willing to trade *y* for *x* (as measured by the MRS) is exactly equal to the rate at which *y* can be transformed into *x* (as measured by the MRT).

Suppose this condition did not hold. Suppose for example that $MRS^1_{xy} < MRT_{xy}$, at a point like A in **Figure 3-48**. In this case, it would be possible to transform *x* into *y* at a rate higher than is needed to keep this person just satisfied with that transformation. That is, we could take some *x* away from her, and give her enough extra *y* to keep her equally happy, and still have some additional *y* left over. We could then give her that additional *y* and make her better off.

We could similarly make this person better off if $MRT_{xy} < MRS^1_{xy}$ (at a point like B in **Figure 3-48**) by transforming *y* in to *x*.

Thus, her utility is maximized if and only if $MRS^1_{xy} = MRT_{xy}$.

If we combine the tangency condition with the PPF equation, then we can in principle solve for the optimal values of Y^{+} and X^{+}. These optimal values are denoted Y^{+*} and X^{+*} in **Figure 3-49**.

Now suppose that we introduce a second person into this economy, and suppose we continue to produce Y^{+*} and X^{+*} as depicted in **Figure 3-49**. We can now ask how these aggregate quantities should be allocated between the two individuals. One possibility is for the original individual to keep consuming everything, but that is just one (extreme) possibility among many.

To think about the other possibilities, it is useful to now imagine a simple exchange economy where the available quantities of the two goods are Y^{+*} and X^{+*}, and to imagine an associated Edgeworth box constructed inside the PPF, as in **Figure 3-50**.

Now imagine constructing the Pareto frontier inside this Edgeworth box, just as we did for our simple exchange economy in Section 3.1. See **Figure 3-51**.

Each point on this Pareto frontier is a Pareto-efficient allocation of Y^{+*} and X^{+*} across the two individuals, and we know that these allocations are characterized by a tangency condition:

$$MRS_{xy}^{1} = MRS_{xy}^{2}$$

Now suppose we pick a particular allocation on the Pareto frontier like A in **Figure 3-51**. We know that for the given amounts of Y^{+*} and X^{+*}, it is not possible to make person 1 better off without making person 2 worse.

However, could we make person 1 better off – without making person 2 worse off – if we chose a different combination of Y^{+} and X^{+} by picking a different point on the PPF, and effectively shift the origin of the Edgeworth box for person 2, as in **Figure 3-52**?

If the answer is yes (as it is in **Figure 3-52**) then the initial point on the PPF cannot be Pareto efficient.

To ensure that person 1 cannot be made better off by moving to a different point on the PPF, while holding utility for person 2 fixed, we must solve the following problem:

$$\max_{x_1, y_1, X^+} u_1(x_1, y_1, l_1) \quad \text{subject to} \quad u_2(x_2, y_2, l_2) = \bar{u}_2$$

$$x_1 + x_2 = X^+$$

$$y_1 + y_2 = Y^{+PE}(X^+)$$

This optimization program states that we maximize the utility of person 1 subject to keeping the utility of person 2 fixed at some level $\bar{u}_2$, and subject to staying on the PPF. (Note that we are still not choosing leisure but we will get to that soon).

Intuitively, the solution to this program must involve two tangencies:

$$MRS_{xy}^1 = MRS_{xy}^2$$

and

$$MRS_{xy}^1 = MRS_{xy}^2 = MRT_{xy}$$

The first tangency implies that no matter what point on the PPF we choose, the allocation between the two people must be on the Pareto frontier corresponding to the Edgeworth box at that point on the frontier.

The second tangency implies that the rate at which either person is willing to trade *y* for *x* (as measured by the MRS) is exactly equal to the rate at which *y* can be transformed into *x* (as measured by the MRT).

This second tangency condition is a straightforward generalization of our tangency condition when we had just one person in the economy. If the condition is not satisfied, then we could make both people better off by transforming *x* into *y* or *y* into *x*, and we would continue to do so until the condition is satisfied.

Figure 3-53 illustrates an allocation, labeled P, where both of these two tangency conditions are satisfied. (The straight lines that are tangent to the PPF and to the ICs are simply there to indicate that all these slopes are equal at the allocation).

It is evident from **Figure 3-53** that the optimality of allocation P is necessarily tied to the distribution of goods between the two people. If we chose a different point on the Pareto frontier passing through P, the slopes of the indifference curves passing through that new point would no longer be equal to the slope of the PFF at $\{X^{+P}, Y^{+P}\}$, and so this point on the PPF would no longer be optimal.

This observation reveals an important point: in general, we cannot separate the question of which point on the PPF we should choose, and how those aggregate quantities are allocated across people. That is, issues of efficiency and distribution cannot be neatly separated.

Generalization to More than Two People

We cannot represent the problem graphically when we have more than two people, but the mathematical representation generalizes quite easily. For example, if we have three people then the optimization program becomes

$$\max_{x_1, y_1, X^+} u_1(x_1, y_1, l_1) \quad \text{subject to} \quad u_2(x_2, y_2, l_2) = \bar{u}_2$$

$$u_3(x_3, y_3, l_3) = \bar{u}_3$$

$$x_1 + x_2 + x_3 = X^+$$

$$y_1 + y_2 + y_3 = Y^{+PE}(X^+)$$

There is now an additional constraint, where the utility of person 3 is fixed at some level $\bar{u}_3$, and the resource constraints now have to account for the allocations to three people.

In general, if there are N people then the optimization program becomes

$$\max_{x_1, y_1, X^+} u_1(x_1, y_1, l_1) \quad \text{subject to} \quad u_i(x_i, y_i, l_i) = \overline{u}_i \quad \text{for} \quad i = 2, \ldots, N$$

$$\sum_{i=1}^{N} x_i = X^+$$

$$\sum_{i=1}^{N} y_i = Y^{+PE}(X^+)$$

This program has $N-1$ utility constraints plus the resource constraints, and at first it might seem daunting. However, if we were to solve the problem using the Lagrange method, we would obtain a straightforward generalization of the tangency conditions we described earlier. In particular,

$$MRS_{xy}^1 = MRS_{xy}^i \quad \forall i$$

$$MRS_{xy}^1 = MRS_{xy}^i = MRT_{xy} \quad \forall i$$

The Pareto frontier in this setting with N people is a surface in $(N+1)$ dimensional space.

Endogenous Leisure

Recall that we have so far abstracted from how we choose the efficient amount of leisure. We noted earlier that if we allocate more potential labour to leisure, then the 2-dimensional PPF is shifted inward, and there is less production available for consumption. This would shrink the dimensions of the Edgeworth box in **Figure 3-53**, and the geometry of the graphical representation starts to get complicated.

The mathematical representation also starts to get complicated. However, we can still make relatively easy progress if we restrict our investigation in the following way.

First, assume that all agents have identical Cobb-Douglas preferences. Second, assume that production is Cobb-Douglas in both sectors, and that $a_1 = a_2$. (Recall that this makes the 2-dimensional PPF linear).

Third, instead of trying to characterize the entire Pareto frontier, suppose we just try to describe the properties of one very special point on that multi-dimensional surface. In particular, suppose we identify the point where all people are allocated exactly the same consumption bundle, in terms of *x*, *y* and *l*. This is called **the symmetric Pareto efficient allocation**.

Since everyone has the same preferences (by assumption), and everyone has the same consumption bundle at the symmetric allocation (by definition), we can identify that special allocation by maximizing the utility of a **representative person**, subject to the aggregate allocation being on the generalized PPF.

Let us examine this optimization program in the context of our numerical example.

Recall that the utility function is

$$u(x, y, l) = x^3 y^7 l^{15}$$

and that the PPF for the numerical example is given by (3.115):

$$Y^{+PE}(X^+) = \frac{(\lambda\tilde{L})^{\frac{3}{4}}(3\tilde{K})^{\frac{1}{4}}}{6} - \frac{X^+}{2}$$

Recall that λ is the fraction of potential labour supplied as actual labour, so the aggregate amount of leisure taken is $(1-\lambda)\tilde{L}$. In the symmetric allocation, everyone gets an equal share of this aggregate leisure, so

$$l = \frac{(1-\lambda)\tilde{L}}{N}$$

for the representative person.

Similarly, everyone receives an equal share of the total amounts of the goods produced, X^+ and Y^+, so

$$x = \frac{X^+}{N}$$

and

$$y = \frac{Y^+}{N}$$

We can now write the maximization program as

$$\max_{X^+, Y^+, \lambda} \quad \left(\frac{X^+}{N}\right)^3 \left(\frac{Y^+}{N}\right)^7 \left(\frac{(1-\lambda)\tilde{L}}{N}\right)^{15}$$

$$\text{subject to} \quad Y^{+PE}(X^+) = \frac{(\lambda\tilde{L})^{\frac{3}{4}}(3\tilde{K})^{\frac{1}{4}}}{6} - \frac{X^+}{2}$$

This is a relatively easy problem to solve, and while we will not work through the details here, the solution has a very simple form:

$$Y^{+PE} = \frac{7(9\tilde{K})^{\frac{1}{4}}\tilde{L}^{\frac{3}{4}}}{180}$$

$$X^{+PE} = \frac{(9\tilde{K}^{\frac{1}{4}})\tilde{L}^{\frac{3}{4}}}{30}$$

$$\lambda^{+PE} = \frac{1}{3}$$

These values might look familiar.

Recall the competitive equilibrium values for the numerical example from Section 3.3-6:

$$C_Y^* = \frac{7(9\tilde{K})^{\frac{1}{4}}\tilde{L}^{\frac{3}{4}}}{180}$$

$$C_X^* = \frac{(9\tilde{K}^{\frac{1}{4}})\tilde{L}^{\frac{3}{4}}}{30}$$

$$S_L^* = \frac{\tilde{L}}{3}$$

The aggregate values in the competitive equilibrium and the aggregate values in the symmetric Pareto efficient allocation are identical.

How is this possible?

3.4-3 THE FIRST WELFARE THEOREM AND THE INVISIBLE HAND

The remarkable result we have just found is a confirmation – in the context of our two-sector exchange economy – of the most important theorem in economics, first proved by Kenneth Arrow and Gerald Debreu (working independently) in 1964.

The First Welfare Theorem

In an economy where

- all agents are price-takers
- there are no increasing returns to scale or other barriers to entry
- there are no public goods
- there are no externalities
- information is symmetric between buyers and sellers,

every competitive equilibrium is Pareto efficient.

This is precisely what we have just discovered about our simple two-sector economy.

The proof of the first welfare theorem is quite technical and is derived in the context of a very general description of the economy. That proof is well beyond the scope of our course.

However, we can get a good sense of how it works, in the context of our two-sector economy. In particular, there are five key properties of the competitive equilibrium that give us the Pareto efficiency result.

Property 1

Recall the cost-minimization condition for a firm in sector *j*:

$$TRS^{j} = \frac{w}{r}$$

Since all firms face the same factor prices, this condition implies that

$$TRS^{X} = TRS^{Y}$$

That is, technical rates of substitution are equated across firms in all sectors. Recall that this is precisely the condition needed to ensure productive efficiency in the allocation of capital and labour across sectors.

Property 2

Recall that free entry drives profit to zero for a firm in sector *j*. This implies that

$$AC_j = p_j$$

Since $MC_j = p_j$, it follows that $MC_j = AC_j$, and this implies that AC_j is minimized. (See **Figure 3-54** for the case of a firm in sector Y). This ensures that we have the right number of firms in each sector.

Taken together, Properties 1 and 2 ensure that the competitive equilibrium is productively efficient.

Property 3

Recall the utility maximization conditions for individual *i*:

$$MRS_{xy}^{i} = \frac{p_X}{p_Y}$$

$$MRS_{ly}^{i} = \frac{w}{p_Y}$$

Since all consumers face the same prices, the first of these conditions implies that

$$MRS_{xy}^{1} = MRS_{xy}^{2} = ... = MRS_{xy}^{N}$$

That is, marginal rates of substitution (between *x* and *y*) are equated across consumers. This ensures that the aggregate quantities of goods produced are allocated efficiently across individuals.

Property 4

Recall the profit-maximization condition for firms in sectors Y and X:

$$MC_Y(y) = p_Y$$

$$MC_X(x) = p_X$$

Since all firms in a given sector face the same output price, these conditions imply that marginal costs are equated across firms in that sector, and this in turn means that we can define marginal cost for the sector as a whole. That is, the marginal cost of producing an extra unit of *y* is MC_Y, and the marginal cost of producing an extra unit of *x* is MC_X.

We can now take the ratio of the two profit-maximization conditions to yield

$$\frac{MC_X(x)}{MC_Y(y)} = \frac{p_X}{p_Y}$$

That is, the ratio of marginal costs is equal to the ratio of output prices.

Now recall that MRT_{xy} effectively measures the opportunity cost of producing an extra unit of good *y* in terms of units of good *x* as we move along the 2-dimensional PPF.

How does this relate to MC_X and MC_Y? Both of these marginal costs are measured in terms of dollars per unit:

$$\frac{\$}{unit_x} \quad \text{and} \quad \frac{\$}{unit_y}$$

respectively.

Thus, the ratio MC_X / MC_Y measures units of *y* in terms of units of *x*. This is precisely what the MRT_{xy} measures. That is,

$$MRT_{xy} = \frac{MC_X(x)}{MC_Y(y)}$$

(It is straightforward to confirm this in the context of our numerical example. If we evaluate $MC_X(x)$ at the equilibrium output level for a firm in sector X, and calculate $MC_Y(y)$ at the equilibrium output level for a firm in sector Y, we will find that the ratio is exactly $\frac{1}{2}$. This is the slope of the 2-dimensional PPF in our numerical example).

We can now relate MRT_{xy} to the equilibrium price ratio.

Since

$$MRT_{xy} = \frac{MC_X(x)}{MC_Y(y)}$$

and

$$\frac{MC_X(x)}{MC_Y(y)} = \frac{p_X}{p_Y}$$

in equilibrium, it follows that

$$MRT_{xy} = \frac{p_X}{p_Y}$$

Taken together with Property 3, this in turn implies that

$$MRT_{xy} = MRS^1_{xy} = MRS^2_{xy} = ... = MRS^N_{xy}$$

That is, the marginal rate of transformation is equated to the marginal rates of substitution. This ensures that the right aggregate quantities of goods are produced; that is, the economy is on the right point on its 2-dimensional PPF.

Note that we can now interpret the slope of the red straight lines in **Figure 3-53** as the (negative) ratio of prices in the competitive equilibrium; see **Figure 3-55**.

Property 5

Recall again the utility maximization conditions for individual *i*:

$$MRS^i_{xy} = \frac{p_X}{p_Y}$$

$$MRS^i_{ly} = \frac{w}{p_Y}$$

Since all consumers face the same wage, the second of these conditions implies that

$$MRS^1_{ly} = MRS^2_{ly} = ... = MRS^N_{ly}$$

That is, marginal rates of substitution between *l* and *y* are equated across consumers. This ensures that the aggregate quantity of leisure is allocated efficiently across individuals.

This condition also ensures that the economy is at the right point on the generalized PPF we depicted in **Figure 3-46**. In particular, recall the PPF for our numerical example, given by (3.115) and repeated below:

$$Y^{+PE}(X^{+}) = \frac{(\lambda\tilde{L})^{\frac{3}{4}}(3\tilde{K})^{\frac{1}{4}}}{6} - \frac{X^{+}}{2}$$

We constructed MRT_{xy} by calculating the slope of $Y^{+PE}(X^{+})$ with respect to X^{+}, but we can also construct an MRT in terms of *l* and *y* by differentiating $Y^{+PE}(X^{+})$ with respect to λ:

$$MRT_{ly} = -\frac{\partial Y^{+PE}(X^{+})}{\partial \lambda} = \frac{(3\tilde{K})^{\frac{1}{4}}}{8(\lambda\tilde{L})^{\frac{1}{4}}}$$

If we set this MRT_{ly} equal to MRS_{ly} evaluated at the equilibrium consumption values, we can solve for $\lambda = \frac{1}{3}$. This is precisely what we found to be the efficient value of λ.

The Invisible Hand

In each of the five properties we just described, *price-taking behaviour* plays a central role. Prices effectively coordinate the independent actions of all the agents in the economy, as if those actions were guided by "an invisible hand" (in the words of Adam Smith).

In the absence of price-taking behaviour (imperfect competition), or when prices are distorted by frictions or incomplete markets, the invisible hand of the market will typically not guide the economy towards Pareto efficiency.

The rest of our course focuses on these distortions and their consequences.

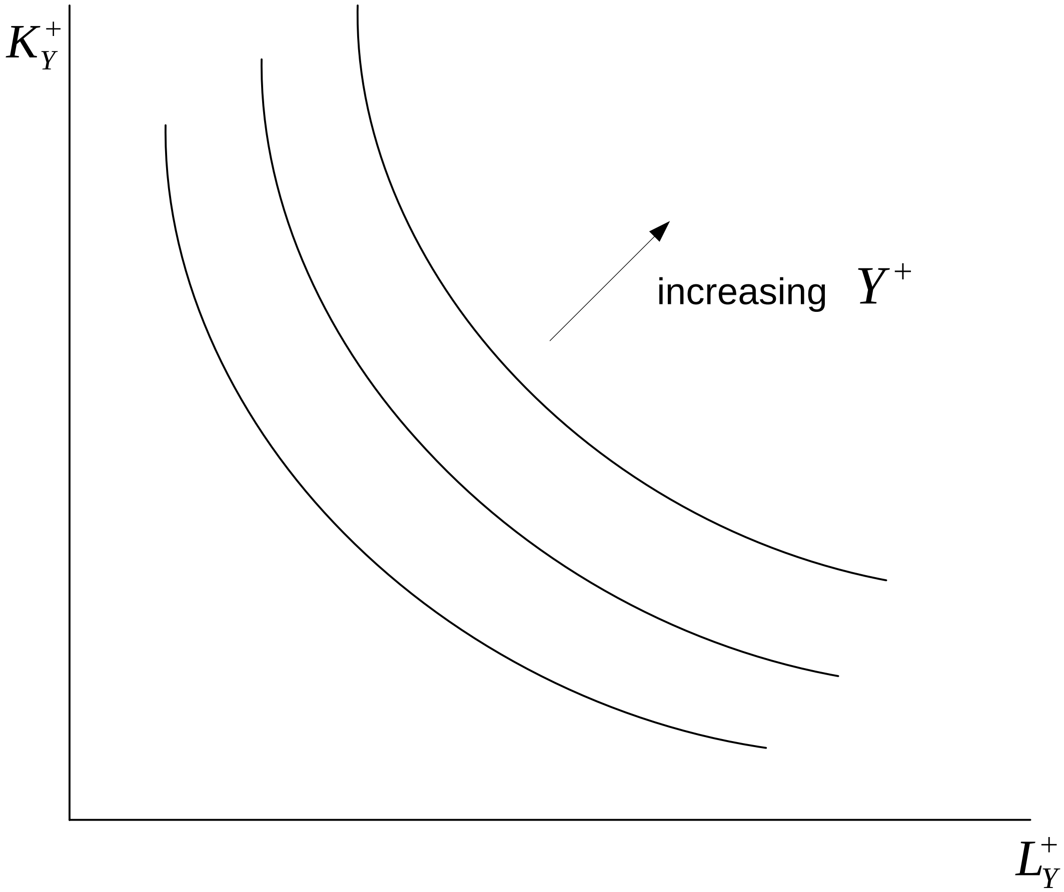


Figure 3-35

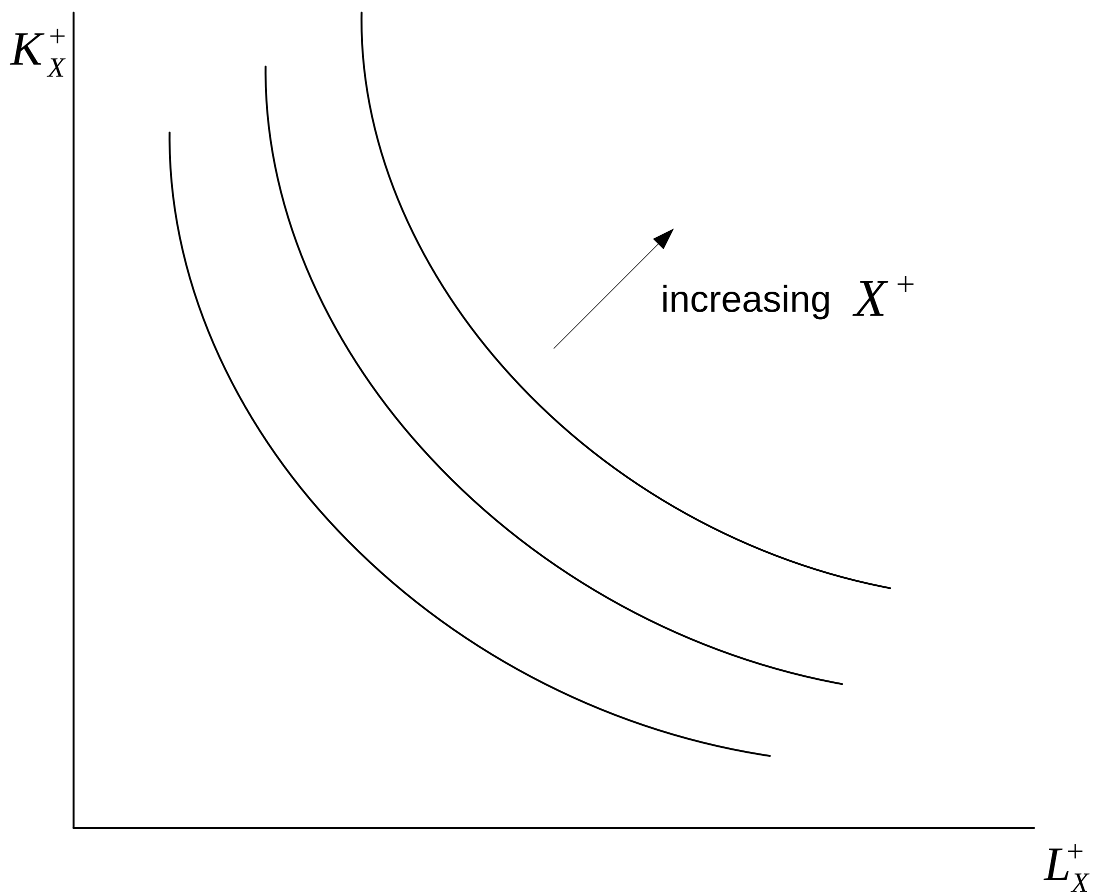
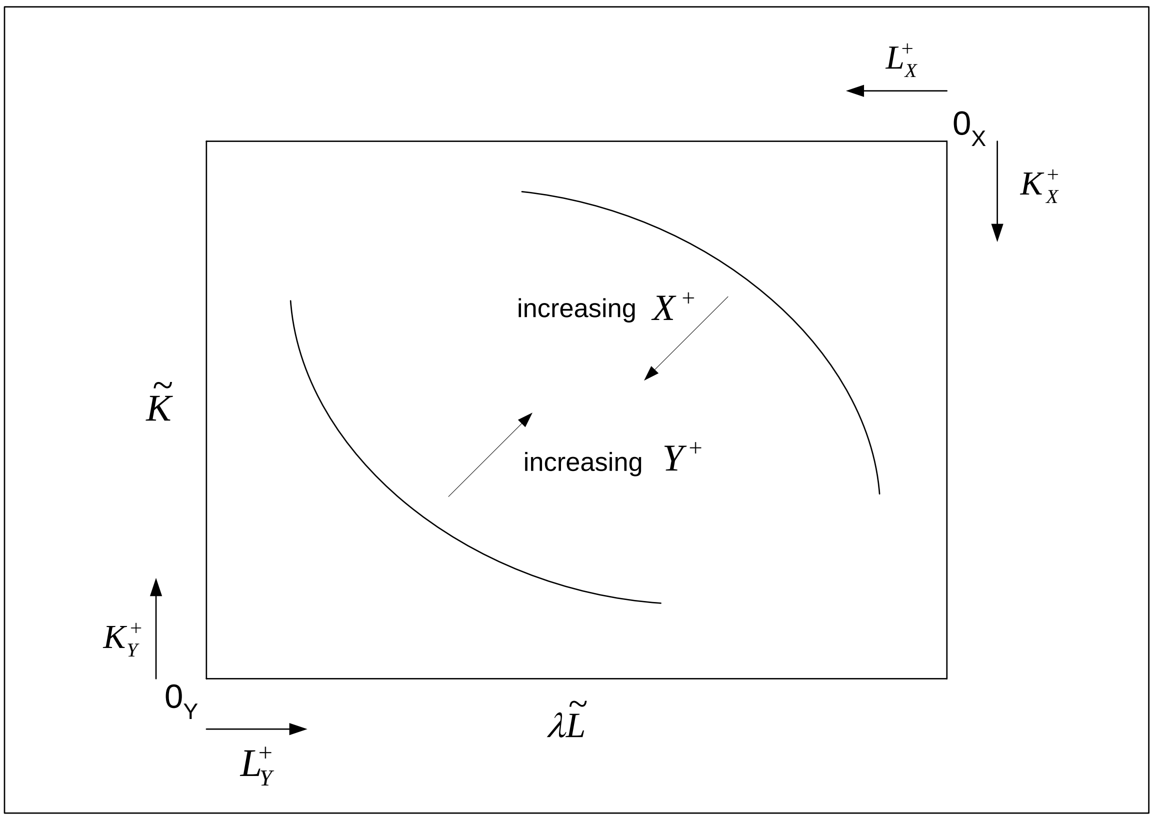


Figure 3-36



L_X^+
0_X
K_X^+
increasing X^+
$\tilde{K}$
increasing Y^+
K_Y^+
0_Y
$\lambda\tilde{L}$
L_Y^+

Figure 3-37

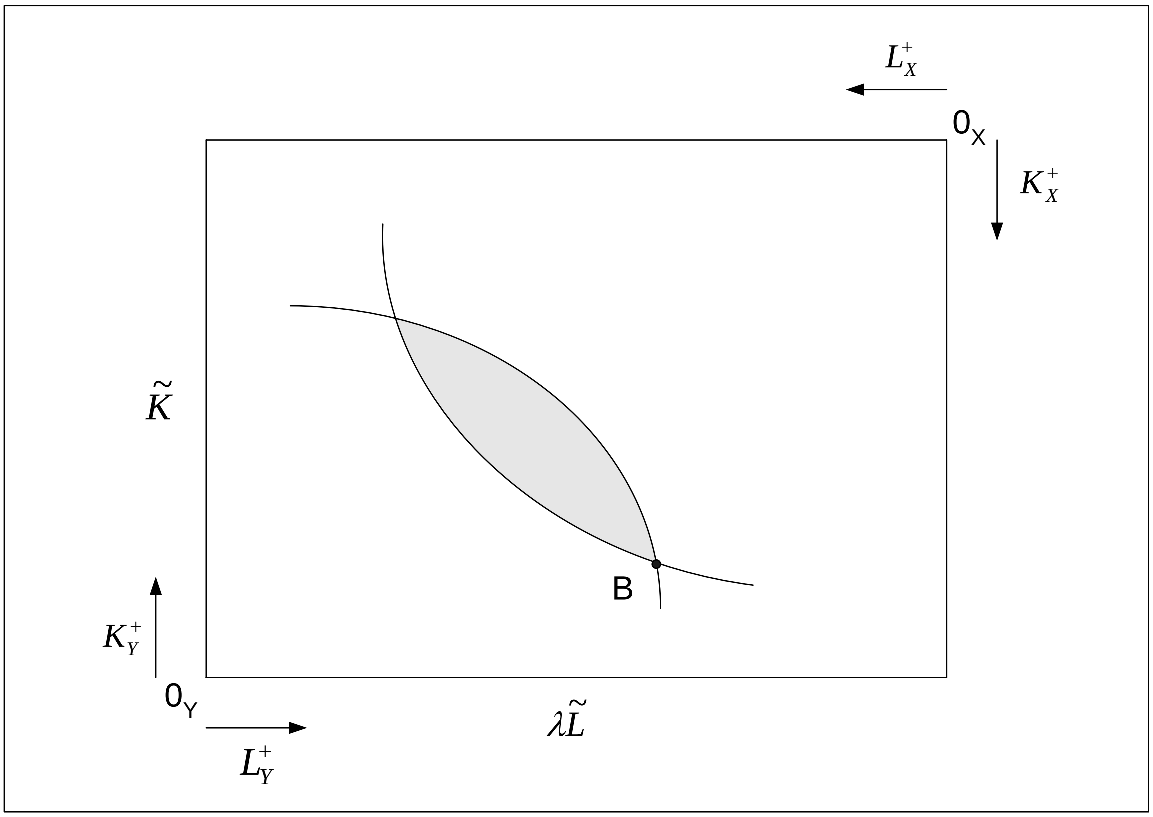
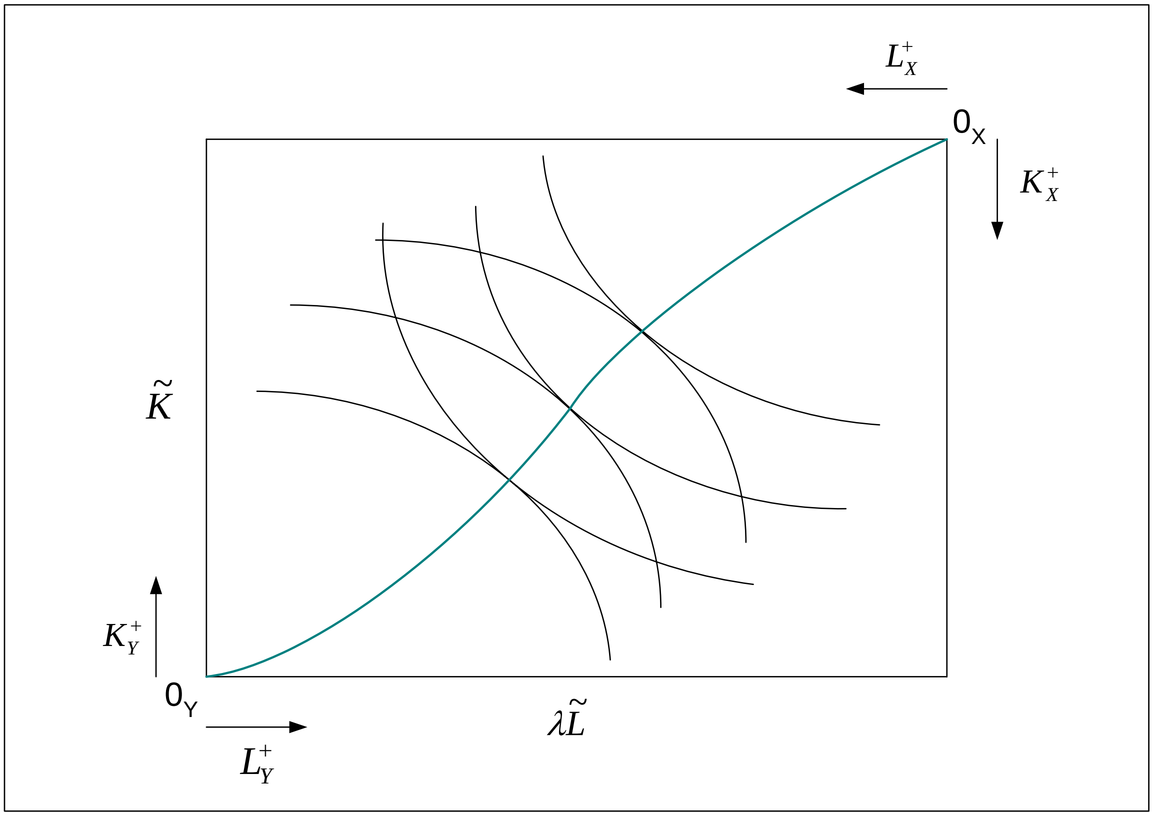


Figure 3-38



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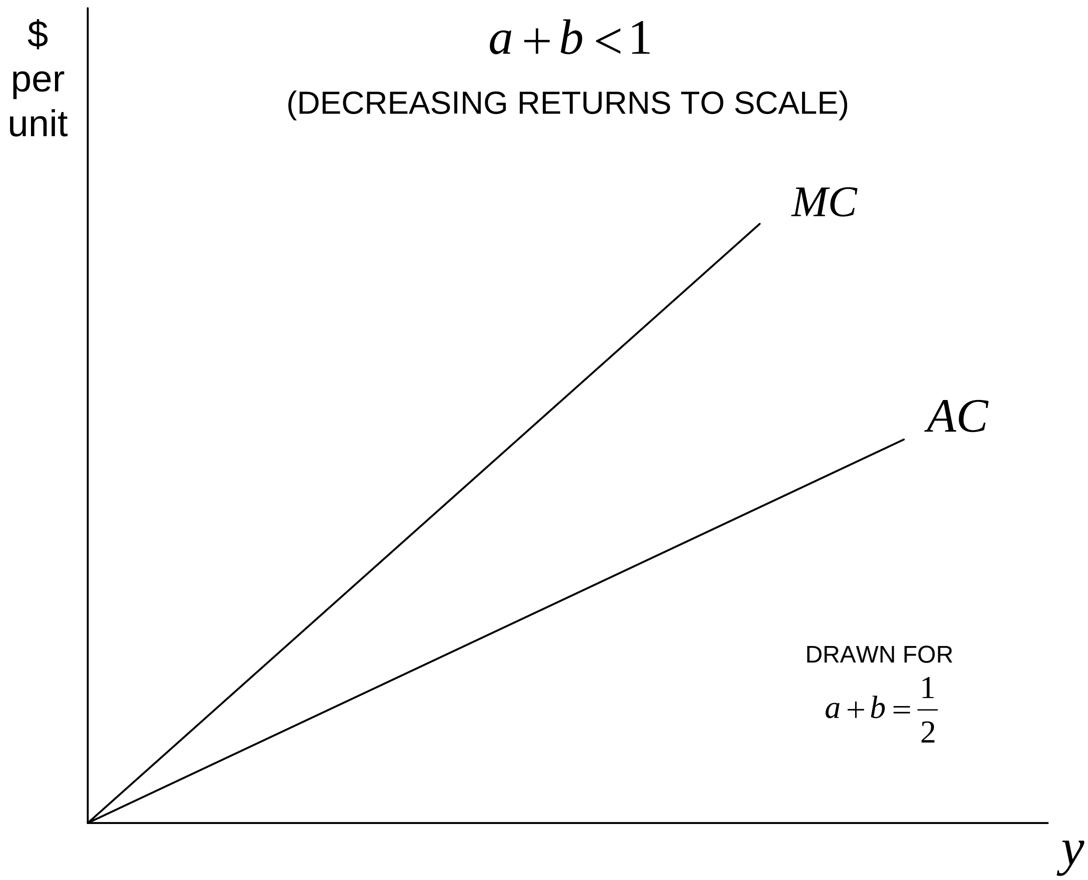
$
per
unit
$a+b<1$
(DECREASING RETURNS TO SCALE)
MC
AC
DRAWN FOR
$a+b=\frac{1}{2}$
y

Figure 3-40

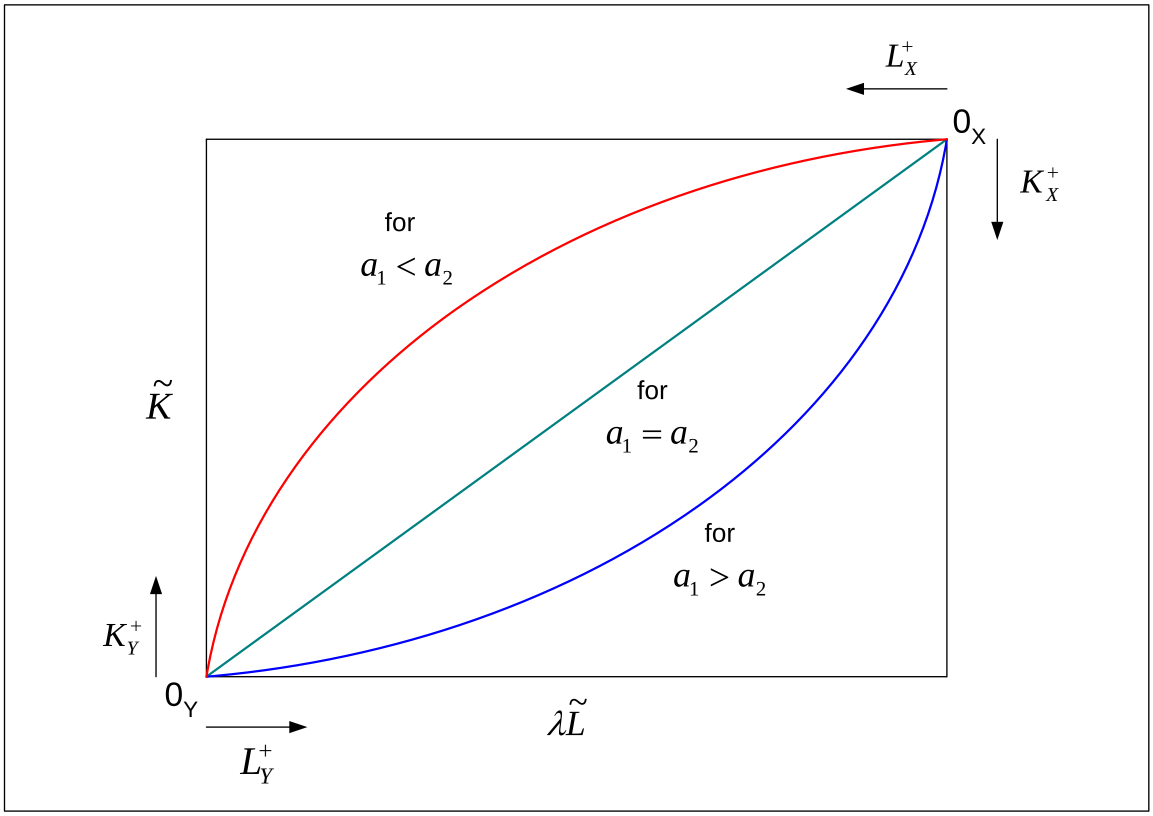
L_X^+
0_X
K_X^+
for
$a_1 < a_2$
$\tilde{K}$
for
$a_1 = a_2$
for
$a_1 > a_2$
K_Y^+
0_Y
$\lambda\tilde{L}$
L_Y^+

Figure 3-41

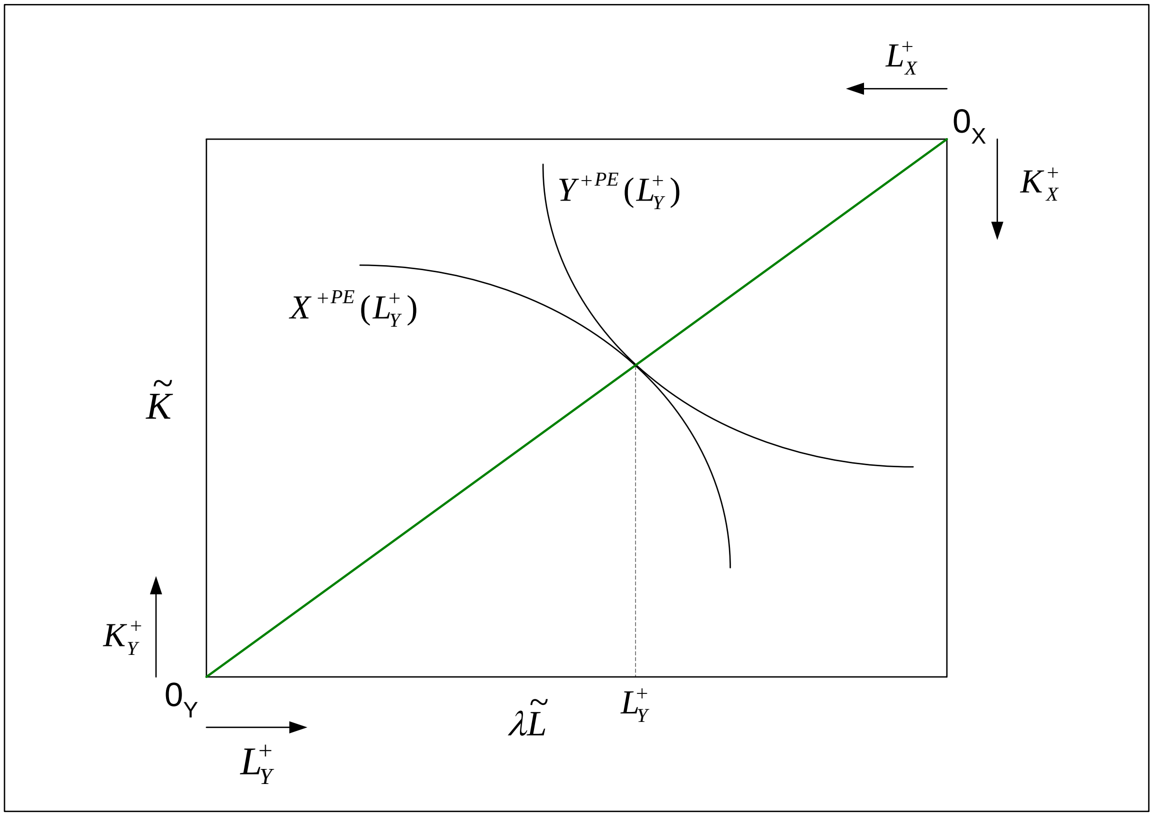

Figure 3-42

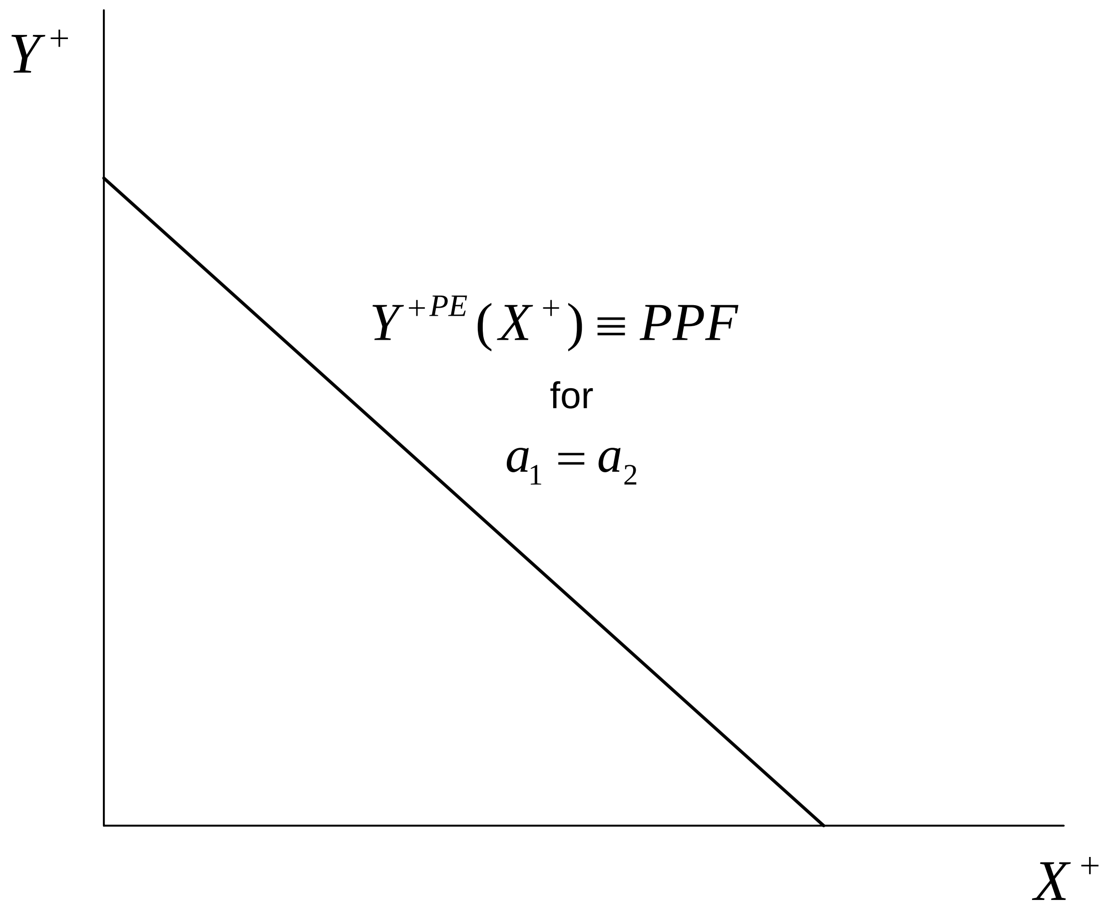

Figure 3-43

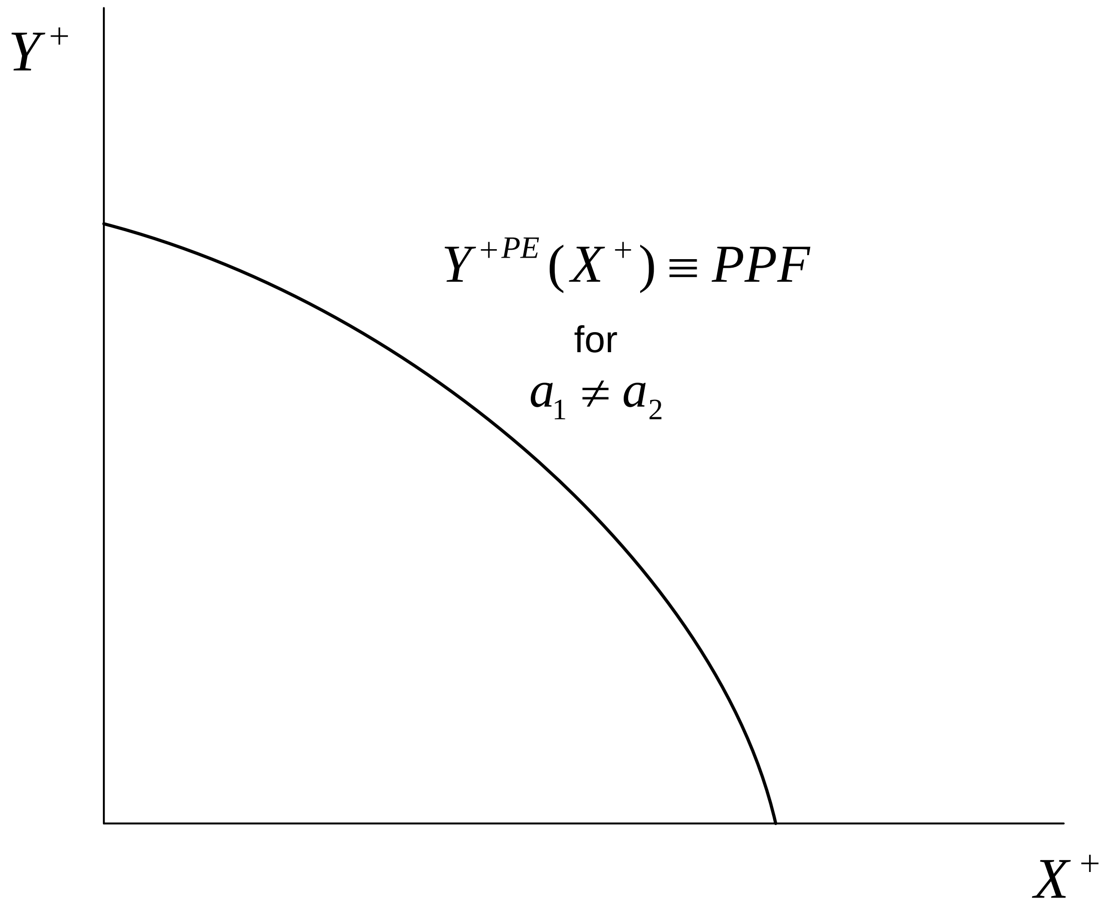

Figure 3-44

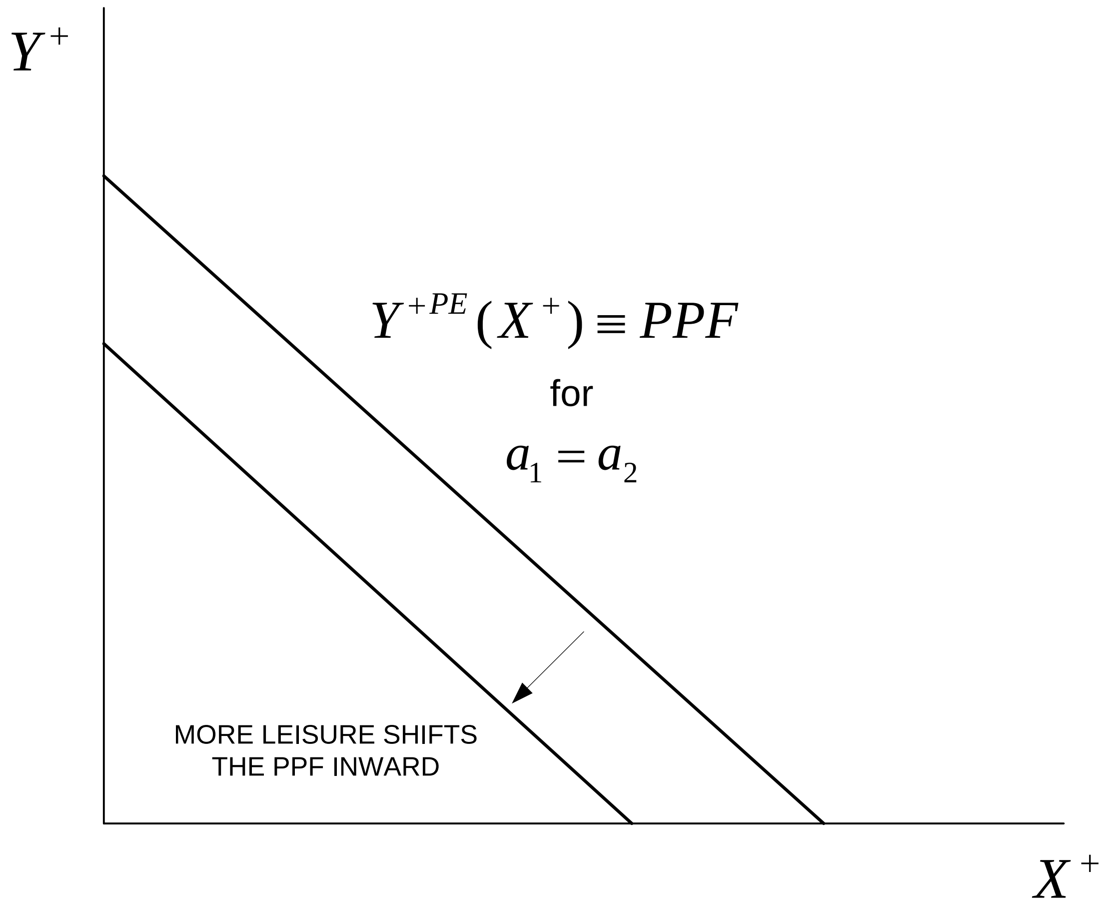
Y^+
$Y^{+PE}(X^+) \equiv PPF$
for
$a_1 = a_2$
MORE LEISURE SHIFTS
THE PPF INWARD
X^+

Figure 3-45

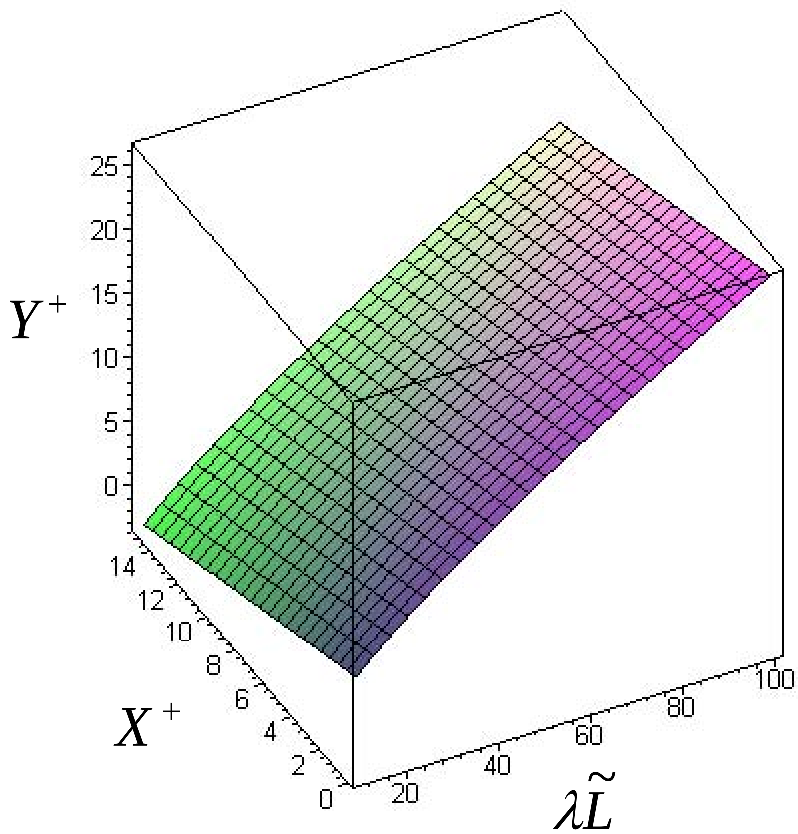

Figure 3-46

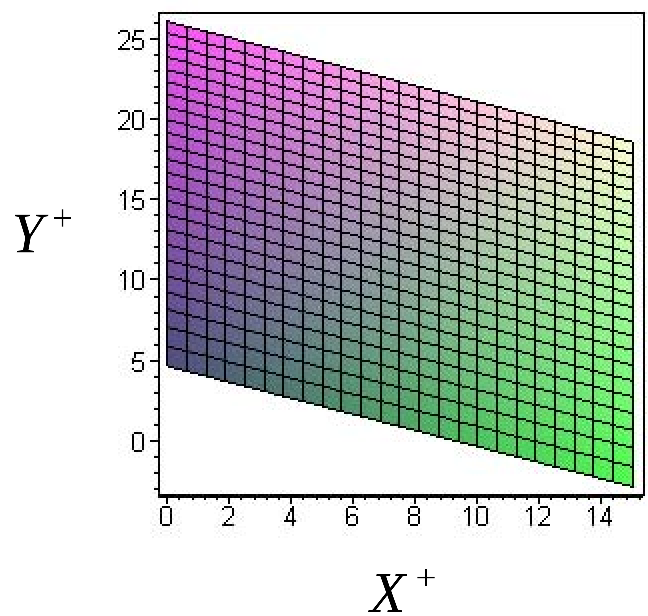

Figure 3-47

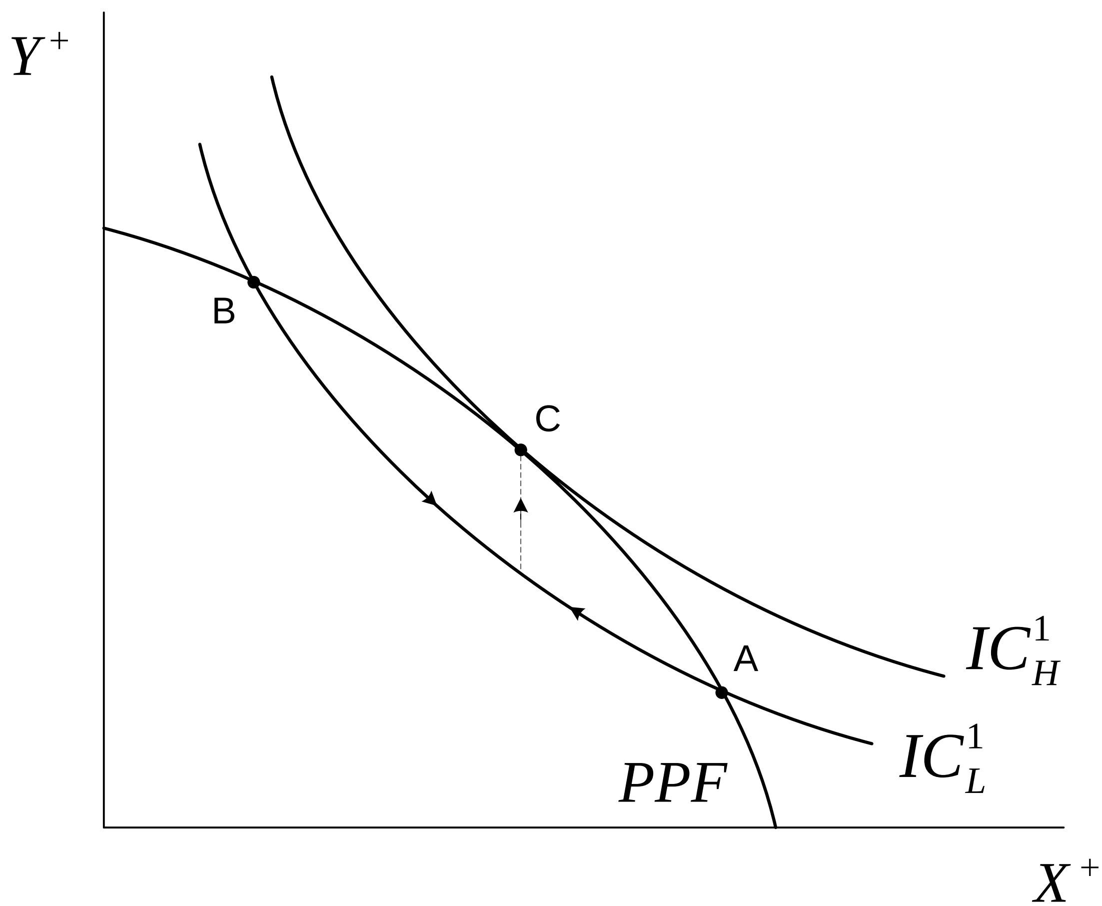

Figure 3-48

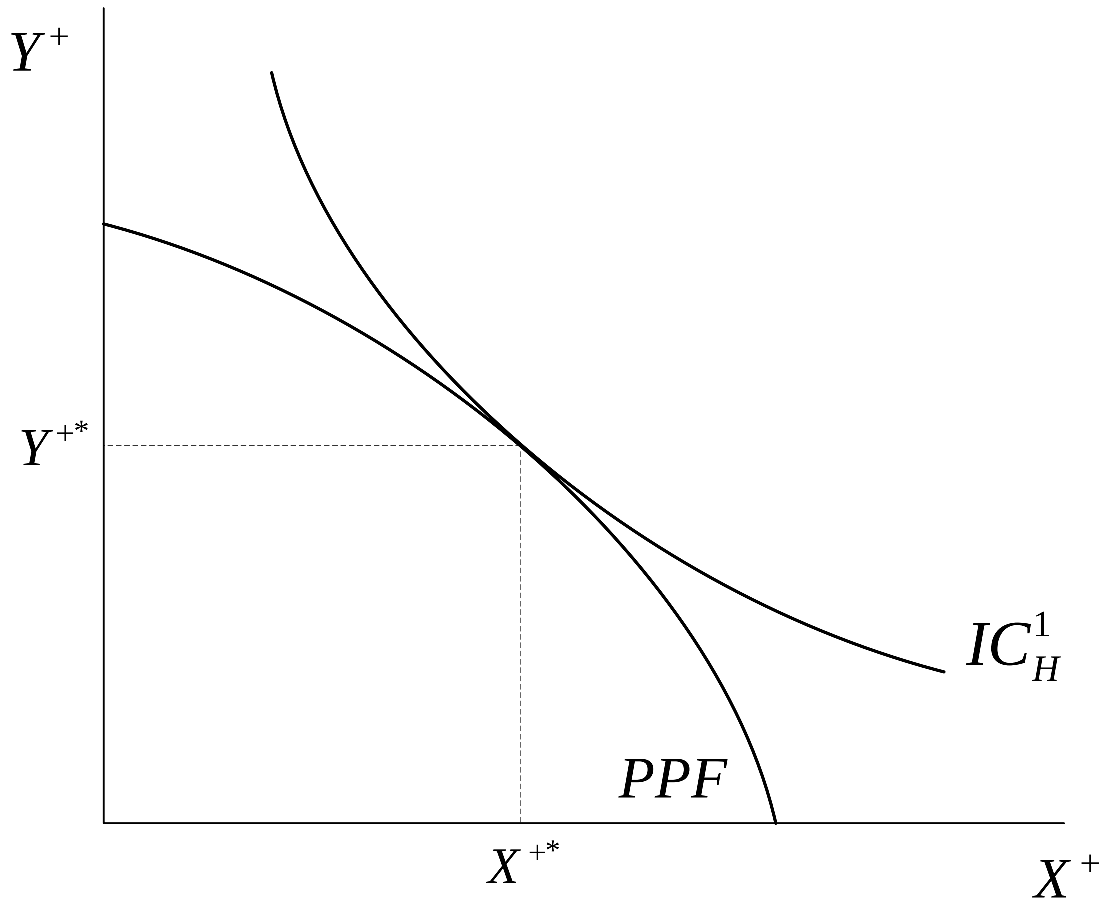

Figure 3-49

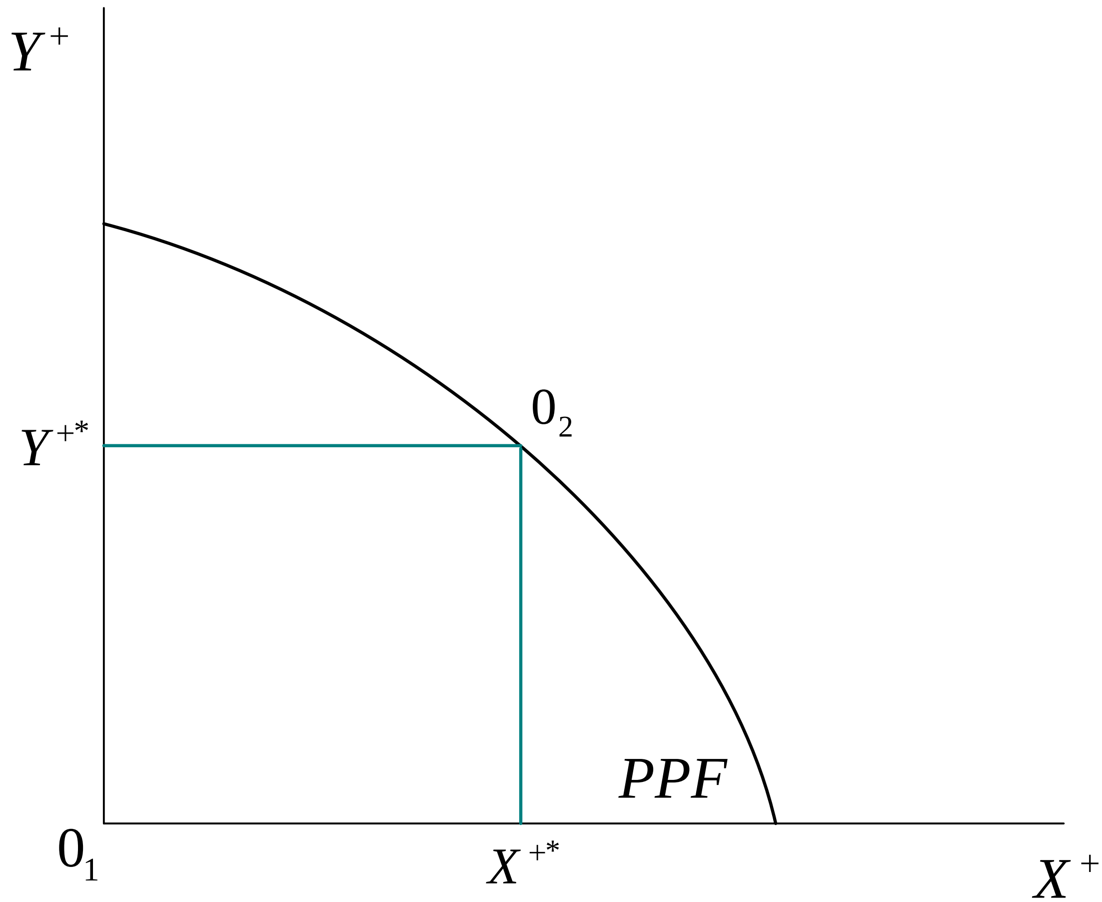

Figure 3-50

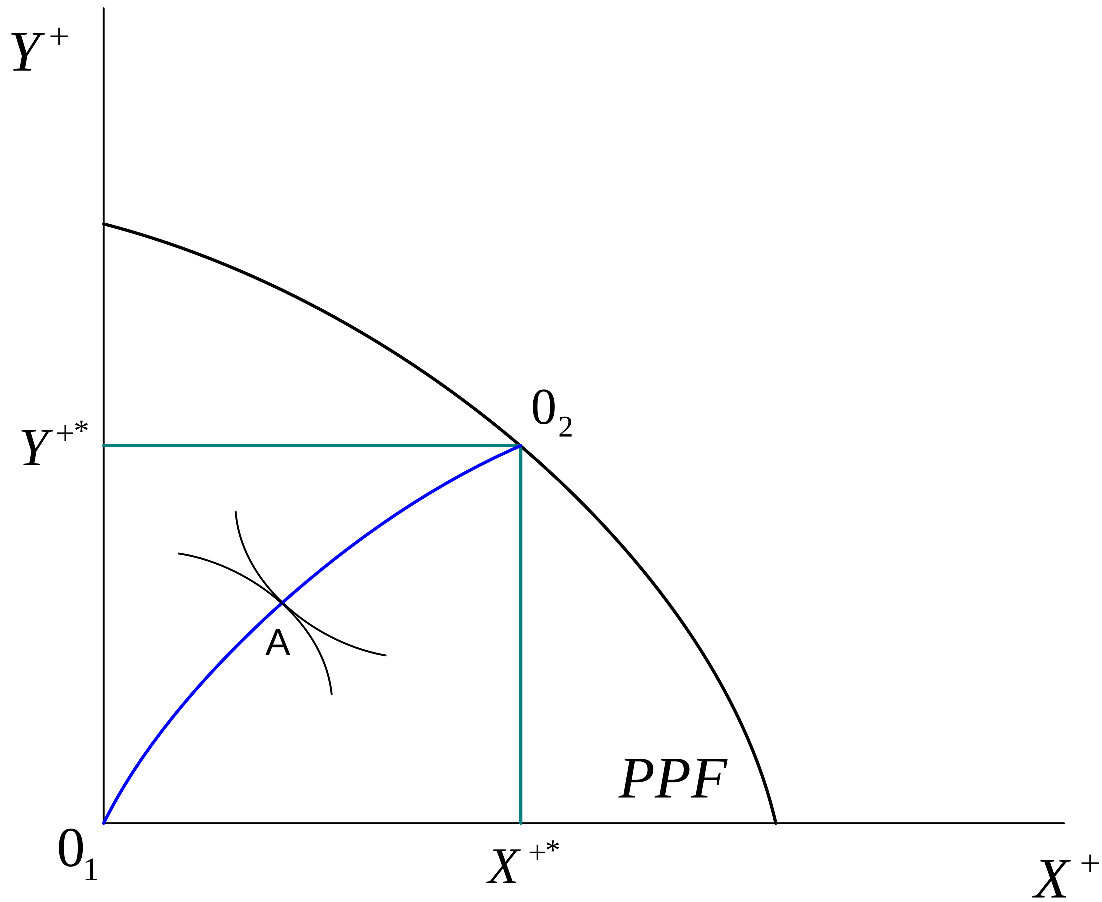

Figure 3-51

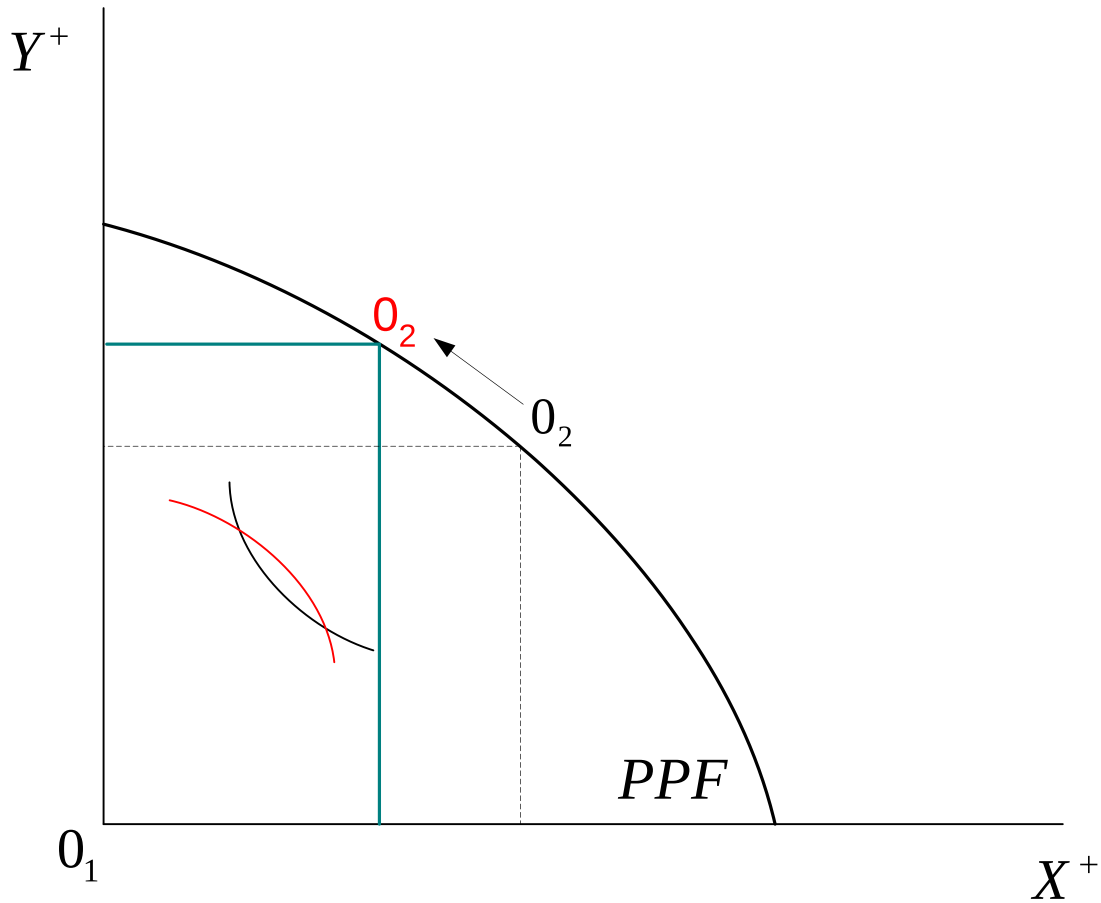

Figure 3-52

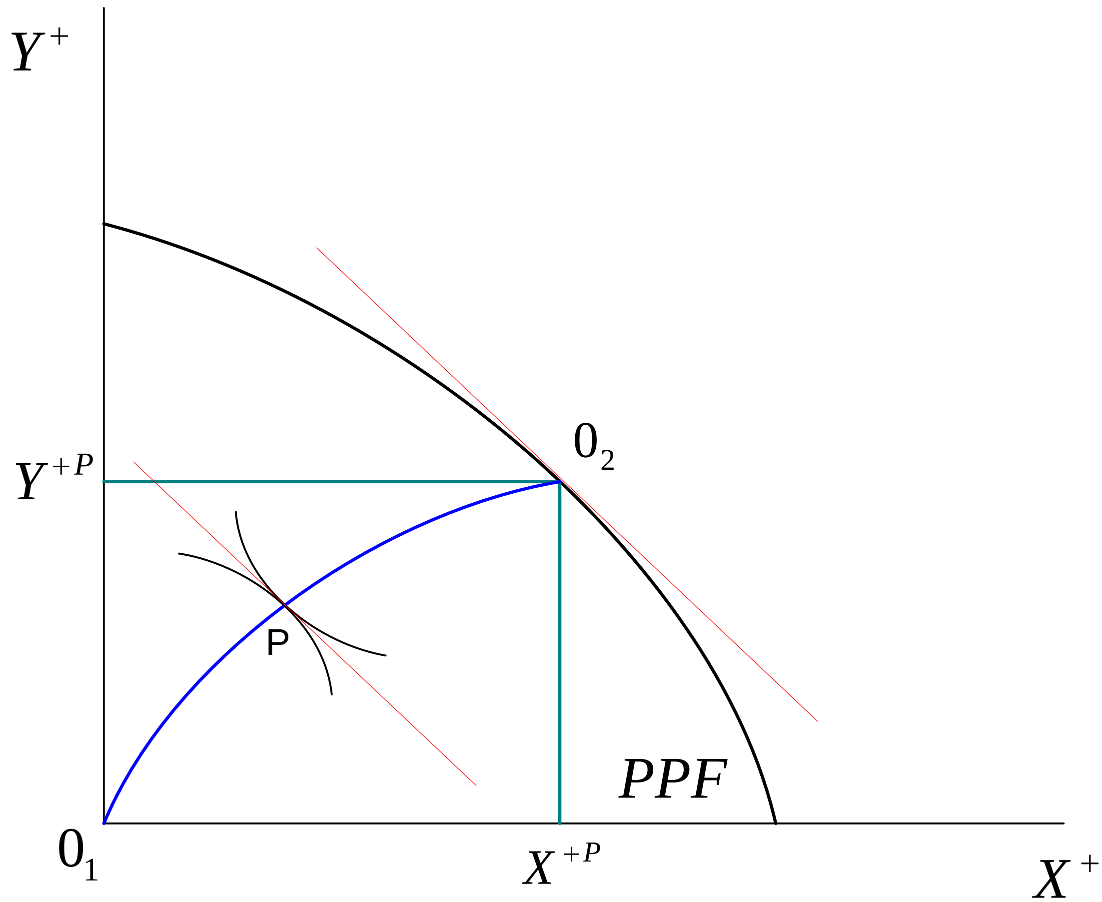

Figure 3-53

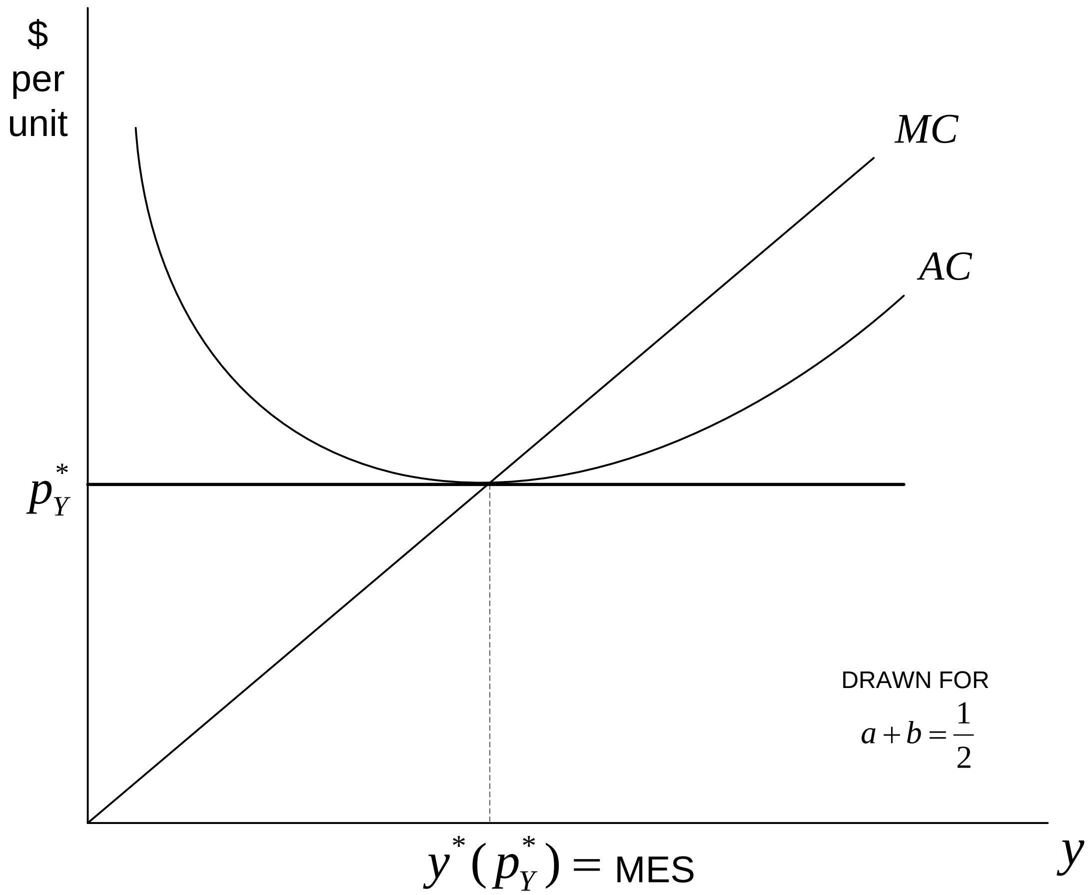

Figure 3-54

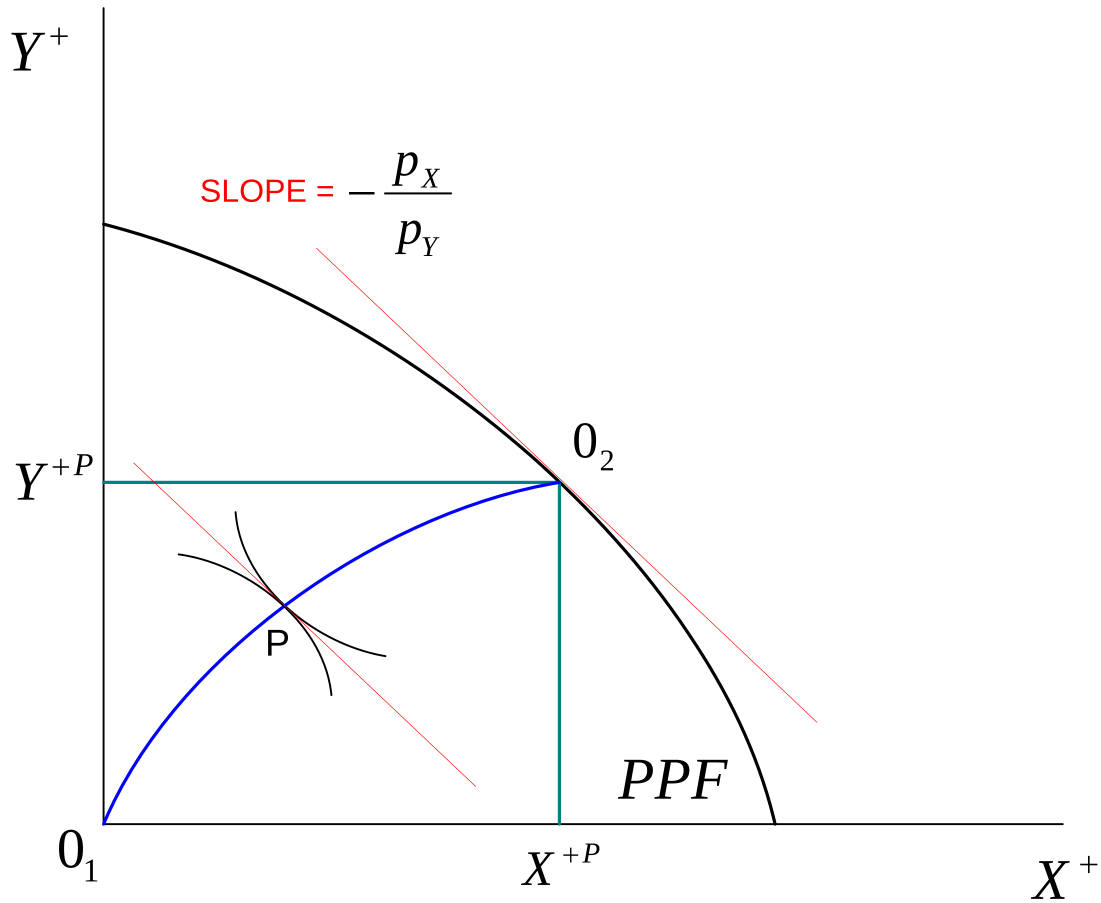

Figure 3-55