5. PUBLIC GOODS

OUTLINE

5.1 INTRODUCTION

Public goods are characterized by two key features:

- joint consumption possibilities
- high exclusion costs

Joint consumption possibilities means that the benefits of the good can be enjoyed by more than one agent at the same time. For example, a park, a lighthouse beam. That is, consumption of public goods is “non-rivalrous” (in contrast to private goods like bread, cheese and wine).

High exclusion costs means that it is costly to prevent agents from consuming the good once it is provided (eg. it is costly to build a fence around a national park or to stop other ships from seeing a lighthouse beam).

Public goods are a type of positive externality in the sense that provision of the good by one agent bestows a positive benefit on other agents (who can enjoy the public good without paying for it).

Note that public goods may or may not be provided by the public sector. Moreover, many goods provided by the public sector are not public goods. Thus, we need to keep a clear distinction between public goods and goods provided by the public sector; they are not the same thing.

Public goods are often classified according to the degree to which they are non-rivalrous and/or non-excludable. In particular:

- **pure public goods** are those that are perfectly non-rivalrous (eg. radio signals, a lighthouse beam, knowledge).
- **impure public goods** (or *congestible public goods*) are subject to congestion; that is, the benefits of consumption declines as more agents use the good (eg. roads, the radio spectrum, a beach).
- **club goods** are congestible public goods with relatively low exclusion costs (eg. a swimming pool, a golf club).

5.2 EFFICIENT PROVISION OF A PURE PUBLIC GOOD

Consider a simple economy with two agents, and two goods *y* and *G*, where *y* is a private good and *G* is a continuous pure public good.

The production possibility frontier (PPF) for this economy is denoted $Y = T(G)$, where we can think of $T(G)$ as a "transformation function". An example is depicted in **Figure 5-1**. The vertical intercept, labeled *M*, corresponds to the maximum amount of the private good that can be produced. This amount can then be "transformed" into *G* via a substitution of resources into the production of *G*. Recall that the slope of this PPF is the marginal rate of transformation (MRT).

We can derive the Pareto frontier for this economy as the solution to the following planning problem:

$$\max_{Y,G,y_1} \; u_1(y_1, G)$$

$$\text{subject to} \quad u_2(y_2, G) = \bar{u}_2$$

$$y_1 + y_2 = Y$$

$$Y = T(G)$$

That is, we are looking for a point on the production possibility frontier, and a division of the associated output of the private good, such that it is not possible to make agent 1 better-off without making agent 2 worse-off.

We can represent the planning problem graphically, in **Figure 5-2**. The upper frame of this figure depicts the PPF and an indifference curve for person 2 corresponding to utility level $\bar{u}_2$. A point on that indifference curve, such as point *A*, represents an allocation to person 2, comprising an amount G^A of the public good, and an amount y_2^A of the private good. We can also indentify a corresponding point on the PPF, labeled A', which tells us the total amount of private good, denoted Y^A, that can be produced when the economy produces an amount G^A of the public good.

What is the allocation to person 1 associated with point *A*? Since the public good is a pure public good, person 1 also receives an amount G^A. The amount of private good allocated to person 1 is the difference between the total amount produced and the amount allocated to person 2. That is, $y_1^A = Y^A - y_2^A$. In the figure, this amount is the vertical distance between A' and *A*.

Every point like *A* on the $u_2 = \bar{u}_2$ indifference curve represents a different allocation of *G* and *y* to person 2, and an associated allocation of *y* to person 1 measured by the vertical distance between the PPF and the indifference curve.

Note that at points *B* and *C*, the allocation of *y* to person 1 is zero, and so allocations on $u_2 = \bar{u}_2$ to the left of *B* or to the right of *C* is are feasible (since person 1 cannot consume a negative amount).

Now suppose we plot the vertical distance between the PFF and the $u_2 = \bar{u}_2$ indifference curve on its own graph, in the lower frame of **Figure 5-2**. This inverted U-shaped function is labeled $CPF_1(\bar{u}_2)$; it is the **consumption possibility frontier** for person 1, conditional on the level of utility corresponding to the $u_2 = \bar{u}_2$ indifference curve. Note that the vertical axis in the lower frame measures y_1. Note too that for every different indifference curve we could draw in the upper frame, there will be a corresponding CPF_1 in the lower frame. It in this sense that the CPF_1 is conditional on $u_2 = \bar{u}_2$.

Allocation *A* in the upper frame of **Figure 5-2** corresponds to point A'' on $CPF_1(\bar{u}_2)$ in the lower frame. This point identifies the allocation to person 1, denoted $\{G^A, y_1^A\}$, when person 2 receives allocation $\{G^A, y_2^A\}$.

Now let us depict an indifference curve for person 1 passing through point A''. This identifies the utility for person 1 when she receives allocation $\{G^A, y_1^A\}$.

We can see immediately from **Figure 5-2** that allocation *A* is not Pareto efficient. There exist alternative allocations on the consumption possibility frontier that would yield higher utility for person 1.

Figure 5-3 identifies the Pareto efficient allocation corresponding to $u_2 = \bar{u}_2$, at point *P*. This allocation is characterized by a tangency between $CPF_1(\bar{u}_2)$ and an indifference curve for person 1; at this solution, it is not possible to make person 1 better without forcing person 2 to a level of utility below $\bar{u}_2$.

Let us summarize how we find this Pareto efficient allocation. We first specify a given level of utility for person 2, corresponding to an indifference curve $u_2 = \bar{u}_2$. We then construct a corresponding $CPF_1(\bar{u}_2)$ for person 1 as the vertical distance between the PPF and the $u_2 = \bar{u}_2$ indifference curve. We then find the highest possible utility for person 1 on this $CPF_1(\bar{u}_2)$, and that corresponds to a tangency between the $CPF_1(\bar{u}_2)$ and an indifference curve for person 1.

What is the mathematical representation of this Pareto-efficient allocation? We know that the last part of the descriptive solution requires a tangency between the $CPF_1(\bar{u}_2)$ and an indifference curve for person 1. That is,

$$\text{slope of } CPF_1(\bar{u}_2) \equiv \frac{\partial CPF_1(\bar{u}_2)}{\partial y_1} = MRS^1_{Gy}$$

What is the slope of $CPF_1(\bar{u}_2)$? Recall that $CPF_1(\bar{u}_2)$ is constructed as the difference between the PPF and an indifference curve for person 2. Thus, the slope of $CPF_1(\bar{u}_2)$ is equal to the difference between the slope of the PPF and the slope of the indifference curve for person 2:

$$\text{slope of } CPF_1(\bar{u}_2) = MRT_{Gy} - MRS^2_{Gy}$$

It follows that the condition for Pareto efficiency is

$$MRT_{Gy} - MRS^2_{Gy} = MRS^1_{Gy}$$

Rearranging this equality yields the **Samuelson condition** for efficient public good provision:

$$MRS^1_{Gy} + MRS^2_{Gy} = MRT_{Gy}$$

The logic of this solution extends to any number of persons, so the generalized Samuelson condition is

$$\sum_{i=1}^{n} MRS^i_{Gy} = MRT_{Gy}$$

That is, an efficient allocation is one where the MRT at that allocation is equal to the sum of the MRSs across all agents at that allocation.

The intuition behind this solution is straightforward: efficiency requires that the marginal cost of providing the public good (in terms of private good foregone) be equal to the *sum* of the marginal private valuations of the public good precisely because *G* is be consumed *jointly* by all agents.

Note the contrast between the Samuelson condition and the requirement for the efficient allocation of private goods, as derived in Topic 3:

$$MRS_{xy}^{1} = MRS_{xy}^{2} = MRT_{xy}$$

In the case of a private good, the marginal unit produced is not consumed jointly; it is consumed by one person or the other. At an efficient allocation, the marginal valuations of the two person should be just equal, and those valuations should in turn be equal to the marginal cost of producing it (in terms of the amount of the other good foregone).

5.3 VOLUNTARY PRIVATE PROVISION OF A PURE PUBLIC GOOD

We can think of voluntary contributions to the provision of a public good as an activity with an associated positive externality: a contribution by any one agent bestows a positive externality on other agents who are also able to consume the public good.

Moreover, the externality is reciprocal in nature: my contribution bestows an external benefit on you; and your contribution bestows an external benefit on me. Accordingly, the best way to analyze the private-provision problem is to use a game-theoretic framework like the one we used in Topic 4.

We will do so in the context of a very simple example.

5.3-1 A Two-Person Public-Good Game

Consider a setting where the PPF for the economy is linear with slope $-\rho$. The transformation function is

(5.1) $$Y = M - \rho G$$

This linearity assumption means that the cost of transforming the private good into the public good is always ρ, regardless of how much G is produced. (This assumption makes the algebra much simpler when we solve the game).

There are two agents in the economy. Agent 1 has utility function

(5.2) $$u_1(G, y_1) = G^{a1} y_1^{b1}$$

and agent 2 has utility function

(5.3) $$u_2(G, y_2) = G^{a2} y_2^{b2}$$

The aggregate amount of private good available for "transformation" into the public good is $M = m_1 + m_2$, where m_1 is the endowment of agent 1, and m_2 is the endowment of agent 2.

5.3-2 Best-Response Functions

The choice problem for agent 2 is

(5.4) $$\max_{g_2, y_2} (g_1 + g_2)^{a2} y_2^{b2} \quad \text{subject to} \quad y_2 = m_2 - \rho g_2$$

where g_2 is her own contribution to the public good, and where g_1 is the contribution by the other agent. Crucially, both agents are able to enjoy the combined contributions, $G = g_1 + g_2$.

Making the substitution for y_2 from the constraint into the objective function, we can rewrite this problem as an unconstrained optimization problem:

(5.5) $$\max_{g_2} (g_1 + g_2)^{a2} (m_2 - \rho g_2)^{b2}$$

This transformed objective function is the payoff function for agent 2, and it will be useful to make that explicit by defining

(5.6) $$v_2(g_1, g_2) = (g_1 + g_2)^{a2} (m_2 - \rho g_2)^{b2}$$

Agent 2 chooses g_2 to maximize this payoff, taking as given the action she expects from agent 1. Setting the first derivative equal to zero and solving for g_2 yields the best-response function for agent 2 (denoted BRF_2):

(5.7) $$g_2(g_1) = \frac{a_2 m_2}{(a_2 + b_2)\rho} - \frac{b_2 g_1}{(a_2 + b_2)}$$

Figure 5-4 plots BRF_2 in (g_1, g_2) space. Note that $g_2(g_1) = g_2^0$ at $g_1 = 0$, where g_2^0 is the sole-agent optimum for agent 2.

Note that $g_2(g_1)$ is decreasing in g_1: the larger the contribution from agent 1, as anticipated by agent 2, the smaller will be the contribution from agent 2. This is the essence of the **free-rider problem** associated with voluntary provision of a public good. Each agent benefits from the contribution made by the other agent, and so each agent has an incentive to "free-ride" on the other agent's contribution and reduce his own contribution accordingly.

In game-theoretic terms, voluntary contributions to the public good are **strategic substitutes**: your contribution is a substitute for mine.

Note that if the contribution from agent 1 is large enough (at $\tilde{g}_1$ in **Figure 5-4**) then agent 2 will contribute nothing at all.

Next consider the choice problem for agent 1 is

(5.8) $$\max_{g_1, y_1} \ (g_1 + g_2)^{a1} y_1^{b1} \quad \text{subject to} \quad y_1 = m_1 - \rho g_1$$

This too can be transformed into an unconstrained optimization problem by making the substitution for y_1 from the constraint into the objective function:

(5.9) $$\max_{g_1} \ (g_1 + g_2)^{a1} (m_1 - \rho g_1)^{b1}$$

This transformed objective function is the payoff function for agent 1, and again it will be useful to make that explicit by defining

(5.10) $$v_1(g_1, g_2) = (g_1 + g_2)^{a1}(m_1 - \rho g_1)^{b1}$$

Agent 1 chooses g_1 to maximize this payoff, taking as given the action she expects from agent 2. Setting the first derivative equal to zero and solving for g_1 yields the best-response function for agent 1 (denoted BRF_1):

(5.11) $$g_1(g_2) = \frac{a_1 m_1}{(a_1 + b_1)\rho} - \frac{b_1 g_2}{(a_1 + b_1)}$$

Figure 5-5 plots BRF_1 in (g_1, g_2) space alongside BRF_2. Note that $g_1(g_2) = g_1^0$ at $g_2 = 0$, where g_1^0 is the sole-agent optimum for agent 1.

5.3-3 The Non-Cooperative Equilibrium

Graphically, the non-cooperative equilibrium (NCE) is the intersection of the best response functions, as depicted in **Figure 5-6**. Algebraically, it is the simultaneous solution of (5.7) and (5.11), which yields

(5.12) $$\hat{g}_1 = \frac{a_1(a_2 + b_2)m_1 - a_2 b_1 m_2}{\rho(a_1(a_2 + b_2) + a_2 b_1)}$$

and

(5.13) $$\hat{g}_2 = \frac{a_2(a_1 + b_1)m_2 - b_2 a_1 m_1}{\rho(a_1(a_2 + b_2) + a_2 b_1)}$$

The aggregate contribution at the NCE is

(5.14) $$\hat{G} = \hat{g}_1 + \hat{g}_2 = \frac{a_1 a_2 (m_1 + m_2)}{\rho(a_1 a_2 + a_1 b_2 + a_2 b_1)}$$

It will also prove useful to calculate consumption of the private good at the NCE. In the case of agent 1,

$$(5.15) \qquad \hat{y}_1 = m_1 - \rho\hat{g}_1 = \frac{a_2 b_1 (m_1 + m_2)}{a_1 a_2 + a_1 b_2 + a_2 b_1}$$

and in the case of agent 2,

$$(5.16) \qquad \hat{y}_2 = m_2 - \rho\hat{g}_2 = \frac{a_1 b_2 (m_1 + m_2)}{a_1 a_2 + a_1 b_2 + a_2 b_1}$$

5.3-4 A Numerical Example

Suppose we assign the following parameter values:

$$\{a_1 = 2, b_1 = 1, a_2 = 1, b_2 = 1, m_1 = 60, m_2 = 60, \rho = \tfrac{3}{4}\}$$

Then the choice problem for agent 2 from (5.5) becomes

$$(5.17) \qquad \max_{g_2} \ (g_1 + g_2)(60 - \frac{3g_2}{4})$$

Setting the derivative with respect to g_2 equal to zero yields:

$$(5.18) \qquad -\frac{3}{4}(g_1 + g_2) + (60 - \frac{3g_2}{4}) = 0$$

Solving this equation for g_2 yields BRF_2:

$$(5.19) \qquad g_2(g_1) = 40 - \frac{g_1}{2}$$

With the assigned parameter values, the choice problem for agent 1 from (5.9) becomes

$$(5.20) \qquad \max_{g_1} \ (g_1 + g_2)^2 (60 - \frac{3g_1}{4})$$

Setting the derivative with respect to g_1 equal to zero yields:

$$(5.21) \qquad -\frac{3}{4}(g_1 + g_2)^2 + 2(g_1 + g_2)(60 - \frac{3g_2}{4}) = 0$$

This is simpler to solve that it might appear. In particular, if we divide both sides by $(g_1 + g_2)$ then it reduces to a linear equation:

(5.22) $$-\frac{3}{4}(g_1 + g_2) + 2(60 - \frac{3g_2}{4}) = 0$$

Solving this equation for g_1 yields BRF_1:

(5.23) $$g_1(g_2) = \frac{160 - g_2}{3}$$

We can now find the NCE as the simultaneous solution to (5.19) and (5.23). Substitute (5.19) for g_2 in (5.23) to obtain

(5.24) $$g_1(g_2) = \frac{160 - \left(40 - \frac{g_1}{2}\right)}{2}$$

and then solve for g_1 to obtain

(5.25) $$\hat{g}_1 = 48$$

Then substitute this solution for g_1 in (5.19) to obtain

(5.26) $$\hat{g}_2 = 16$$

The aggregate contribution is

(5.27) $$\hat{G} = \hat{g}_1 + \hat{g}_2 = 64$$

and the private good consumption is

(5.28) $$\hat{y}_1 = m_1 - \rho\hat{g}_1 = 60 - \left(\frac{3}{4}\right)48 = 24$$

for agent 1, and

(5.29) $$\hat{y}_2 = m_2 - \rho\hat{g}_2 = 60 - \left(\frac{3}{4}\right)16 = 48$$

for agent 2.

5.3-5 Isopayoff Contours

Recall from Topic 4 that we can usefully construct isopayoff contours to illustrate some key properties of the NCE in relation to the Pareto frontier. (An isopayoff contour is a locus of points in the strategy space along which the payoff is constant).

Figure 5.7 depicts a set of isopayoff contours for agent 2, together with BRF_2. The payoff to this agent rises as we move along BRF_2 toward $\tilde{g}_1$. That is, the payoff to agent 2 is highest when agent 1 contributes so much that agent 2 reduces his contribution to zero. Conversely, the payoff to agent 2 is lowest when agent 1 contributes nothing at all (at the sole-agent optimum for agent 2).

Why? Contributions to the public good made by agent 1 directly benefit agent 2 precisely because it is a public good.

Note that this property of the public good game is in stark contrast to what we found in the reciprocal negative externality game from Topic 4. In that setting, the activity of another agent was detrimental to any one agent because of the associated external cost. In that setting, the payoff to an agent was highest at the sole-agent optimum.

Figure 5.8 depicts a set of isopayoff contours for agent 1, together with BRF_1. The payoff to this agent rises as we move along BRF_1 away from g_1^0. That is, the payoff to agent 1 is lowest at her sole-agent optimum (where agent 2 contributes nothing).

Note from **Figure 5.8** that BRF_1 passes through the flat spots of the isopayoff contours for agent 1. This reflects the fact that each point on BRF_1 represents a tangency between a horizontal line (corresponding to a particular value of g_2) and the highest possible isopayoff contour for agent 1. (Similarly, BRF_2 in **Figure 5-7** passes through the vertical points of the isopayoff contours for agent 2).

5.3-6 The Pareto Frontier

The Pareto frontier in this game can be derived in a now familiar way: we maximize the payoff to one agent subject to maintaining a given payoff to the other agent. If we use the preference parameter values from the numerical example, we can derive a very simple closed-form solution for the Pareto frontier, so we will focus on that case.

It makes no difference whether we maximize $v_1(g_1, g_2)$ and hold $v_2(g_1, g_2)$ constant, or *vice versa*. Here we will maximize $v_2(g_1, g_2)$. Thus, our planning problem is

(5.30) $$\max_{g_1, g_2} \ (g_1 + g_2)(m_2 - \rho g_2)$$

$$\text{subject to} \ \ (g_1 + g_2)^2 (m_1 - \rho g_1) = \bar{v}_1$$

We will not work through the algebra here, but this problem can be solved for a Pareto frontier in (g_1, g_2) space, given by

(5.31) $$g_2^{PF}(g_1) = \frac{2m_1 + m_2 - 3\rho g_1}{2\rho}$$

This Pareto frontier is depicted in **Figure 5-9**. The Pareto frontier – labeled *PF* in the figure – is the locus of tangencies of the isopayoff contours for the two agents. In this example, the frontier is linear.

The logic of this tangency-based solution is the same as that underlying the derivation of the Pareto frontier in the exchange economy from Topic 2, and the Pareto frontier in the reciprocal externality game from Topic 4.

In particular, if we hold $v_1(g_1, g_2)$ fixed at some value $\bar{v}_1$, corresponding to a particular isopayoff contour for agent 1, and then maximize $v_2(g_1, g_2)$, then the solution is a point of tangency between an isopayoff contour for agent 2 and the isopayoff contour for agent 1 corresponding to $v_1(g_1, g_2) = \bar{v}_1$. As we vary the value at which $v_1(g_1, g_2)$ is fixed, we trace out a continuum of such tangency points. That continuum is the Pareto frontier.

Note from **Figure 5-9** that the Pareto frontier is not anchored at the sole-agent choices, g_1^0 and g_2^0. (This is in contrast to negative externality-setting we examined in Topic 4). We will return to this point later.

Figure 5-10 overlays the Pareto frontier on the best-response functions and the corresponding NCE. The key message from this figure is that the *NCE is inefficient*; it lies below the Pareto frontier.

We can interpret this result from two different perspectives. The first is to think of the public good problem in terms of a positive externality. Each agent ignores the benefit that her contribution to the public good bestows on the other agent precisely because that benefit is external. This external benefit is nonetheless part of the true social benefit of a contribution, and efficiency requires that it be taken into account. From this perspective, a public good problem is just a special case of a reciprocal positive externality.

The second perspective is to think in terms of the aforementioned free-rider problem. Each agent benefits from the contribution made by the other agent, and so each agent has an incentive to "free-ride" on the other agent's contribution and reduce his own contribution accordingly. Recall that we interpreted the negative slope of the BRF from this "free-rider" perspective.

These two different perspectives are simply different ways of describing the same thing. When one agent free rides, she fails to take into account the impact that her reduced contribution has on the other agent (via the foregone external benefit) and that is why the free-riding creates inefficiency.

5.3-7 The Core

Recall from Topic 4 that the **core** with respect to the NCE in the reciprocal externality game is the set of Pareto efficient allocations that Pareto-dominate the NCE.

The core for the public good game is highlighted in **Figure 5-11**, which overlays on **Figure 5-10** the isopayoff contours passing through the NCE. These contours correspond to the payoffs at the NCE (and accordingly, they are labeled $\hat{v}_1$ and $\hat{v}_2$ for agents 1 and 2 respectively).

The core is the highlighted segment of the Pareto frontier that lies within the shaded lens-shaped region bounded by the two-isopayoff contours. (Recall that this region is the *region of mutual benefit* because it constitutes the set of points that Pareto-dominate the NCE).

These concepts are the same as those we have seen before in the context of the simple exchange economy in Topic 2, and in the context of the reciprocal externality game from Topic 4.

5.3-8 The Samuelson Condition Revisited

How does the Pareto frontier in the game-theoretic setting relate to the Samuelson condition from Section 5.2? They are both descriptions of Pareto efficiency, so we should be able to find a correspondence between them.

We will begin by deriving the Samuelson condition in the context of our simple two-person economy, with the preference parameter values from the numerical example; that is, at $\{a_1 = 2, b_1 = 1, a_2 = 1, b_2 = 1\}$.

Recall the basic properties of our economy. The utility function for person 1 is

(5.32) $$u_1(G, y_1) = G^2 y_1$$

and we know that the marginal rate of substitution for this Cobb-Douglas function is

(5.33) $$MRS^1_{Gy} = \frac{a_1 y_1}{b_1 G} = \frac{2y_1}{G}$$

The utility function for person 2 is

(5.34) $$u_2(G, y_2) = Gy_1$$

and the marginal rate of substitution for this function is

(5.35) $$MRS^2_{Gy} = \frac{a_2 y_2}{b_2 G} = \frac{y_2}{G}$$

The marginal rate of transformation for our economy is $MRT_{Gy} = \rho$.

Now recall the Samuelson condition from Section 5.1:

(5.36) $$MRS^1_{Gy} + MRS^2_{Gy} = MRT_{Gy}$$

Thus, in the context of our example, the Samuelson condition is

(5.37) $$\frac{2y_1}{G} + \frac{y_2}{G} = \rho$$

We also know that an efficient allocation must satisfy the resource constraints, and in our example economy these constraints are

(5.38) $$y_1 = m_1 - \rho g_1$$

(5.39) $$y_2 = m_2 - \rho g_2$$

(5.40) $$G = g_1 + g_2$$

Making these substitutions in (5.37) and solving for g_2 yields

(5.41) $$g_2^{SC}(g_1) = \frac{2m_1 + m_2 - 3\rho g_1}{2\rho}$$

where "SC" indicates that these $\{g_1, g_2\}$ pairs satisfy the Samuelson condition.

Now compare this result with our description of the Pareto frontier from the public-good game, repeated here as

(5.42) $$g_2^{PF}(g_1) = \frac{2m_1 + m_2 - 3\rho g_1}{2\rho}$$

Comparing (5.42) with (5.41) tells us that the Samuelson condition and the game-theoretic Pareto frontier are one in the same.

5.3-9 The NCE Revisited

Given that we can relate the Pareto frontier to the Samuelson condition, can we also relate the NCE to a relationship between MRT_{Gy}, MRS^1_{Gy} and MRS^2_{Gy}?

To do so, let us first evaluate the MRS for each agent at the NCE. For agent 1, this is

(5.43) $$M\hat{R}S^1_{Gy} = \frac{a_1 \hat{y}_1}{b_1 \hat{G}}$$

Making the substitutions for $\hat{y}_1$ and $\hat{G}$ from (5.15) and (5.14), this reduces to

(5.44) $$M\hat{R}S^1_{Gy} = \rho$$

For agent 2, the MRS at the NCE is

(5.45) $$M\hat{R}S^2_{Gy} = \frac{a_2 \hat{y}_2}{b_2 \hat{G}}$$

Making the substitutions for $\hat{y}_2$ and $\hat{G}$ from (5.16) and (5.14), this also reduces to

(5.46) $$M\hat{R}S^2_{Gy} = \rho$$

Thus, the NCE is characterized by the following condition tangency:

(5.47) $$M\hat{R}S^1_{Gy} = M\hat{R}S^2_{Gy} = \rho = MRT_{Gy}$$

This condition should look familiar from Topic 3. Recall from that topic that in an economy with two private goods *x* and *y*, the competitive equilibrium <u>and</u> allocative efficiency were both characterized by

(5.48) $$MRS^1_{xy} = MRS^2_{xy} = MRT_{xy}$$

The NCE in the public-good setting is characterized by an analogous condition because both agents effectively treat the public good like a private good: neither agent takes into account the benefit his own contribution bestows on the other agent; he accounts only for the private benefit he enjoys. In contrast, Pareto efficiency requires that those external benefits be taken into account, and hence the social valuation is the sum of the private valuations.

There is one final link we can make between the NCE and our description of the public good problem from section 5.2. Recall **Figure 5-3**. It depicts the Pareto-efficient allocation corresponding to a particular fixed level of utility for agent 2. There was nothing special about that particular utility level, and for every different utility level there is a different Pareto-efficient allocation.

Now let us focus on one particular Pareto-efficient allocation. Suppose we specify the fixed utility level for agent 2 as the utility he receives in the NCE, denoted $\hat{u}_2$. This is depicted in the upper frame of **Figure 5-12**. We can identify the NCE in this figure because we earlier learned that the NCE is characterized by

(5.49) $$M\hat{R}S^2_{Gy} = MRT_{Gy}$$

This equality of slopes occurs where the vertical distance between the PPF and the indifference curve corresponding to $u_2 = \hat{u}_2$ is at its greatest, at the points labeled E and E' in **Figure 5-12**. Thus, these points identify the NCE values for y_2, G and Y.

We can now construct the associated consumption possibility frontier for agent 1, labeled $CPF_1(\hat{u}_2)$ in the lower frame of **Figure 5-12**. The NCE consumption point for agent 1 is at the point labeled E'' because we know that $\hat{y}_1 = \hat{Y} - \hat{y}_2$.

Figure 5-13 overlays on **Figure 5-12** the Pareto-efficient allocation corresponding to $u_2 = \hat{u}_2$, indentified as allocation Q in the figure. Note that the Pareto-efficient allocation involves a higher value of G, and lower values of y for both agents, than at the NCE.

Crucially, allocation *Q* also involves a strictly higher level of utility for agent 1 than he obtains at the NCE, but agent 2 is no worse off than at the NCE (by construction). Thus, allocation *Q* is in the core with respect to the NCE.

We can now draw a clear link between our Samuelson-condition figures and our game-theoretic figures: allocation *Q* in **Figure 5-13** corresponds to a special point on the boundary of the core from **Figure 5-11**, as highlighted in **Figure 5-14**.

Now suppose we re-specify the planning problem that underlies **Figure 5-13**, and maximize the utility of agent 2. Specifically, let us maximize the utility of agent 2 subject to holding the utility of agent 1 fixed at his NCE utility level:

(5.50) $$\max_{Y,G,y_2} \; u_1(y_2, G)$$

$$\text{subject to} \quad u_1(y_1, G) = \hat{u}_1$$

$$y_1 + y_2 = Y$$

$$Y = T(G)$$

The solution to this problem identifies allocation *R* in **Figure 5-15**, and this corresponds to point *R* on the boundary of the core in **Figure 5-16**.

Generally, points *Q* and *R* will be different points on the PPF; the aggregate level of the public good will not be the same at all points on the Pareto frontier. Only in special cases will the Pareto-efficient value of *G* be unique.

In our example setting, one such special case is where both agents have the same preference parameters. In that case, it can be shown that the unique Pareto-efficient value of *G* is

(5.51) $$G^* = \frac{aM}{\rho(a+b)}$$

where $M = m_1 + m_2$.

Moreover, this result extends to a setting with *n* agents, where

(5.52) $$M = \sum_{i=1}^{n} m_i$$

It can also be shown that in this special setting with *n* agents whose preference parameters are identical, the NCE level of *G* is

(5.53) $$\hat{G} = \frac{aM}{\rho(a + nb)}$$

Two points stand out when comparing G^* and $\hat{G}$. First, if $n = 1$ then $\hat{G} = G^*$. That is, if there is only one agent then the NCE is Pareto-efficient because there is no external benefit (or equivalently, there is no one on which this single agent can free-ride).

Second, consider the ratio

(5.54) $$\frac{\hat{G}}{G^*} = \frac{a + b}{a + nb}$$

This ratio gets smaller as *n* rises. This reflects a conjecture first made by economist Mancur Olson (1932 – 1966): free riding gets worse as the population grows.

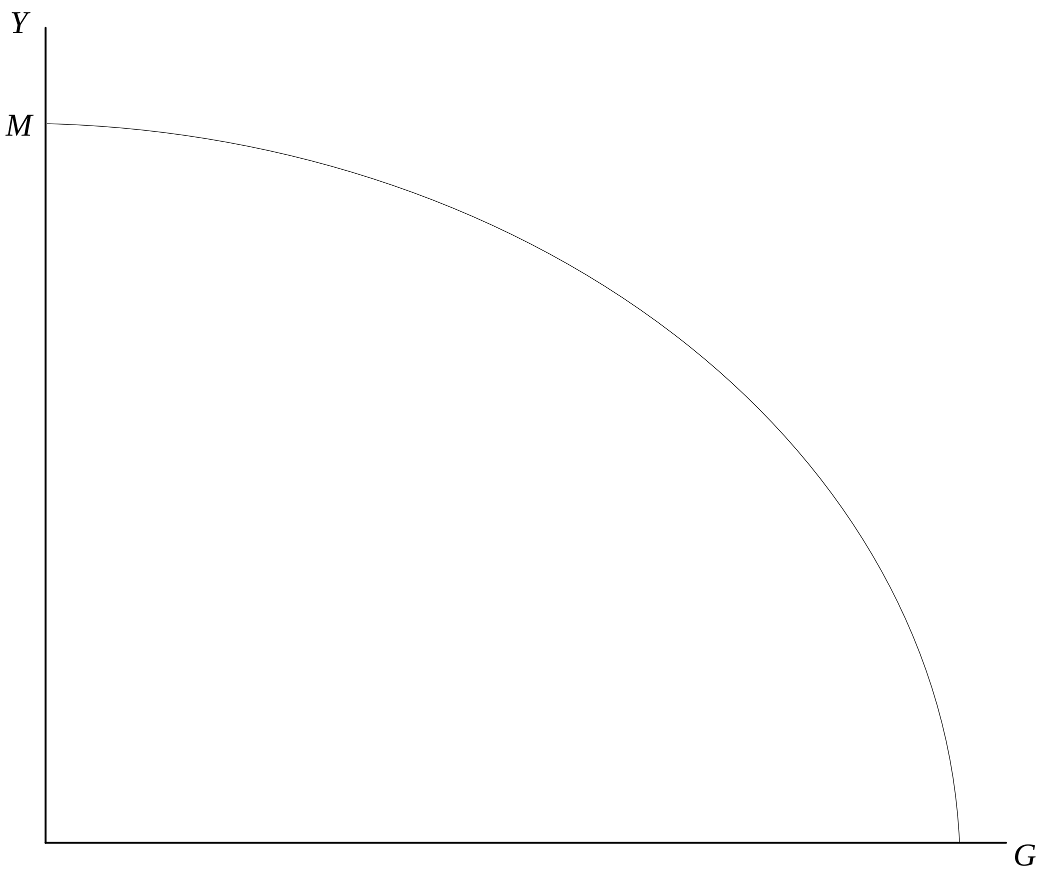

Figure 5-1

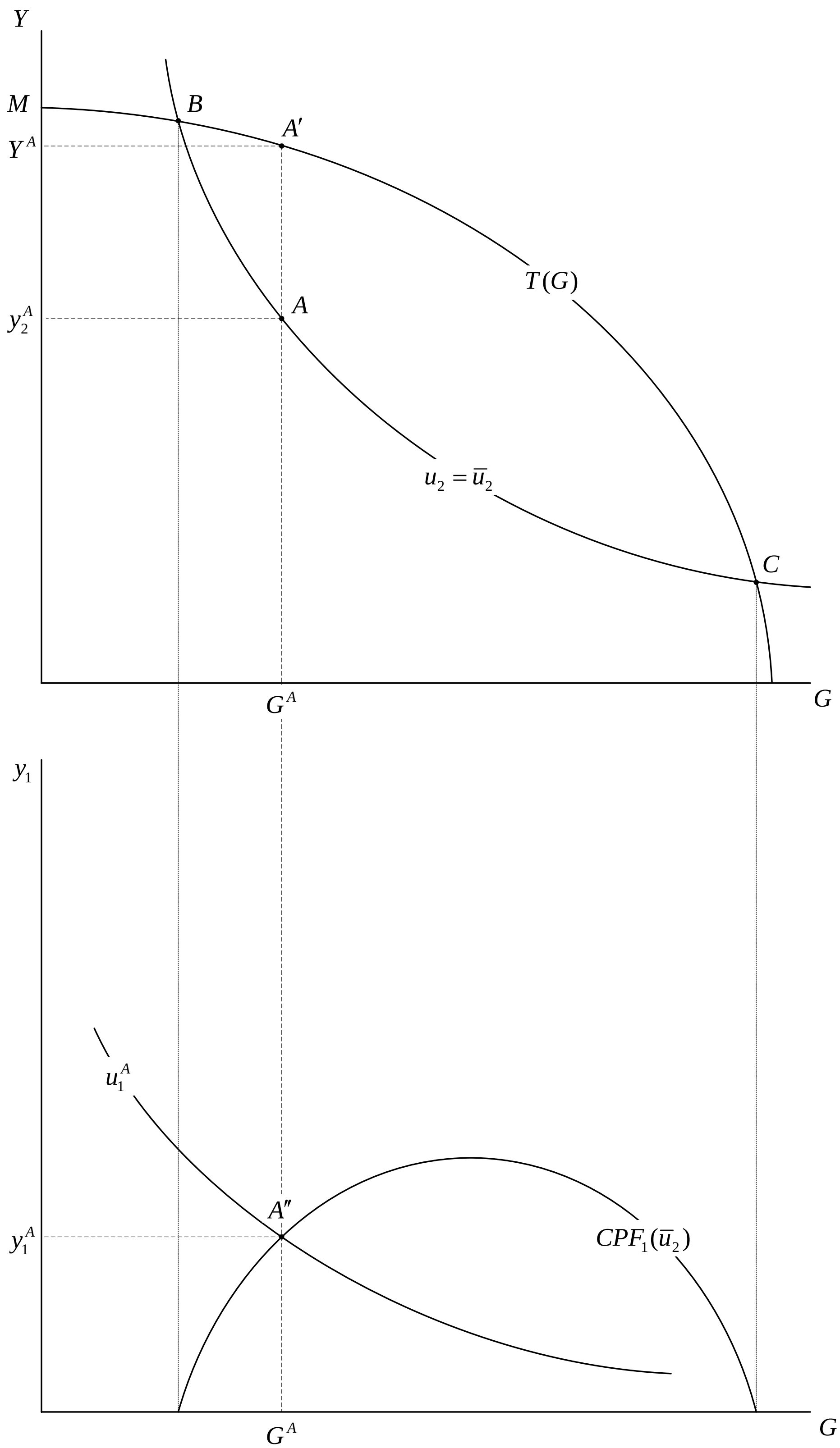

Figure 5-2

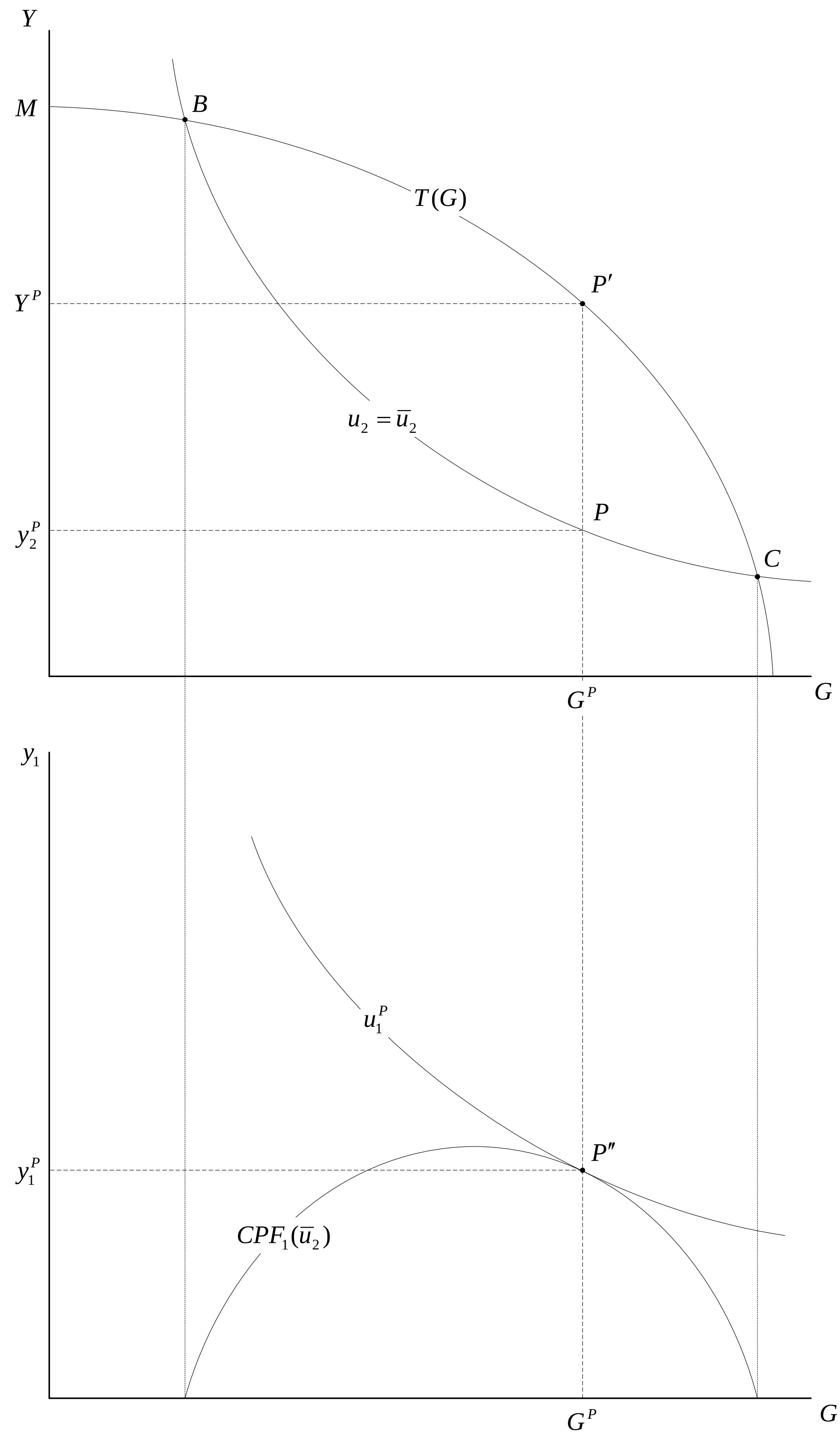

Figure 5-3

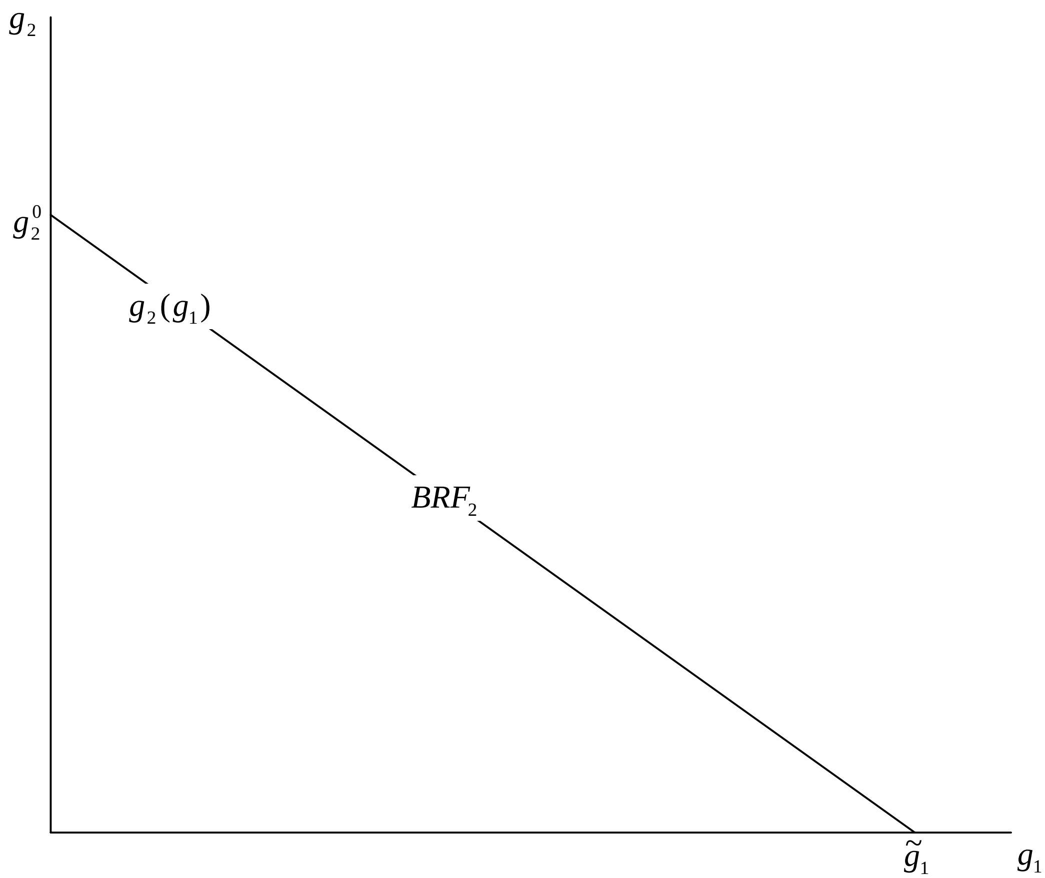

Figure 5-4

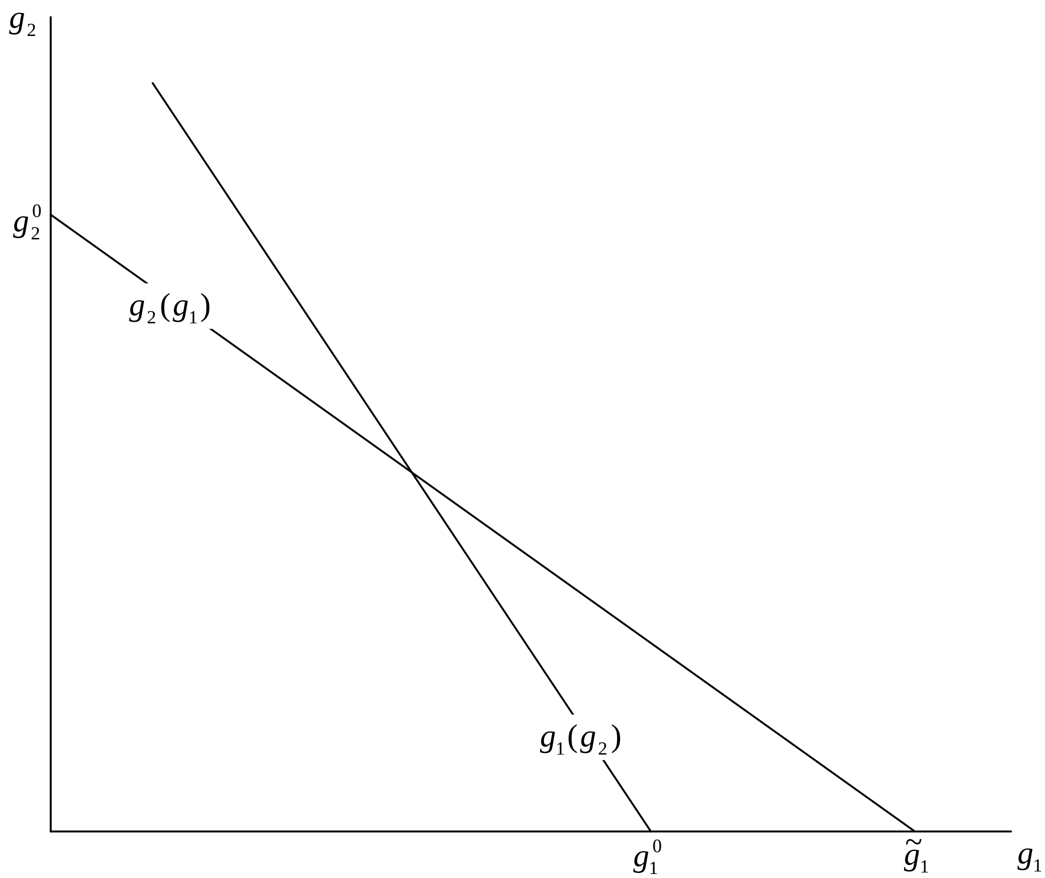

Figure 5-5

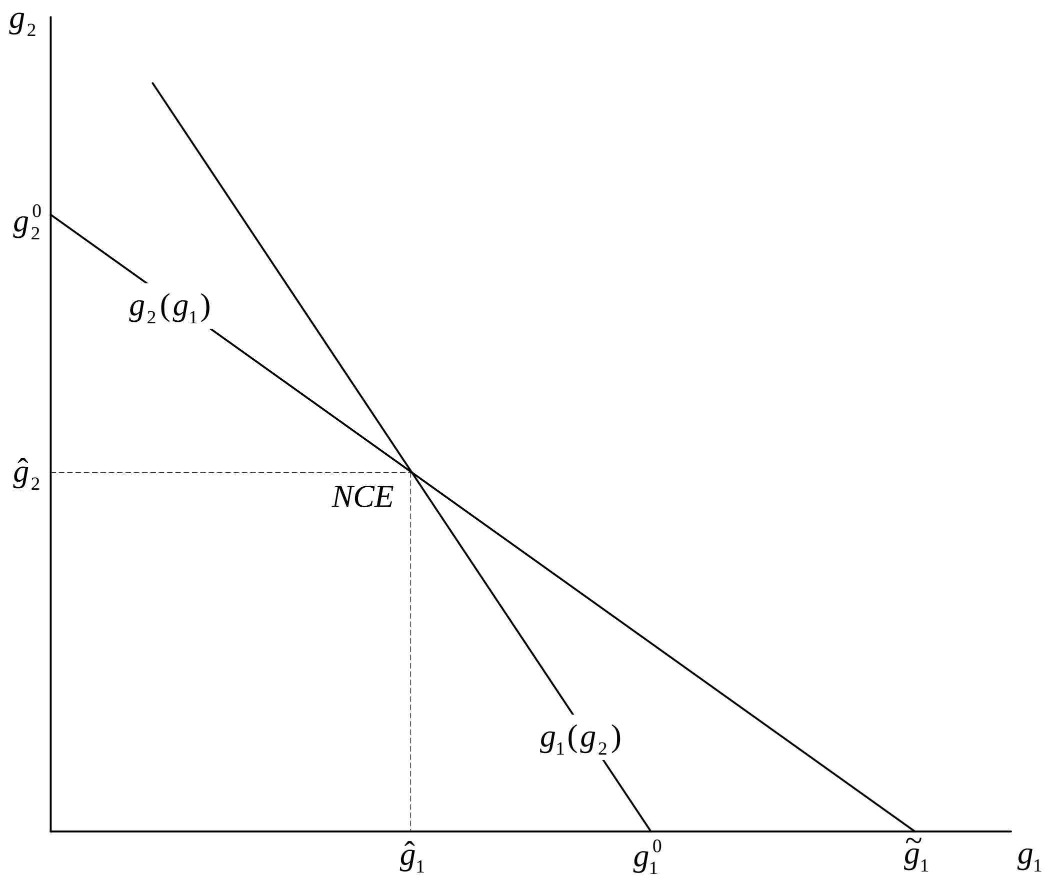

Figure 5-6

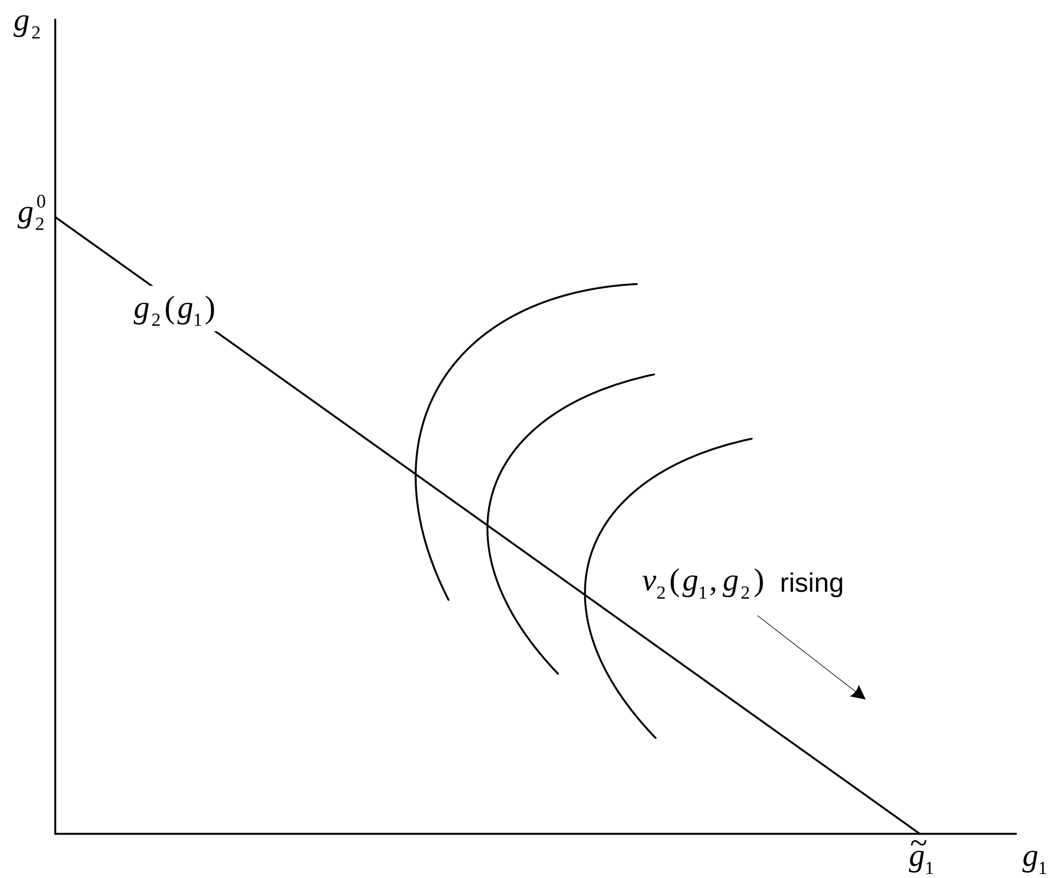

Figure 5-7

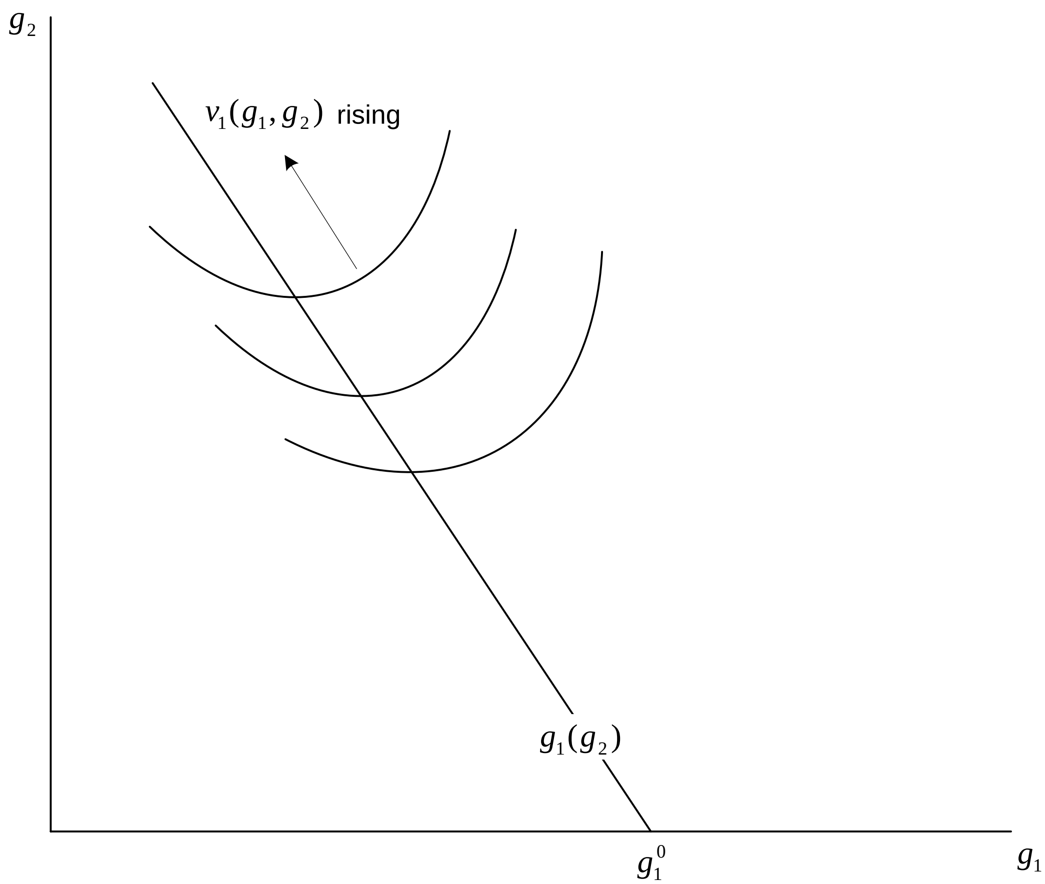

Figure 5-8

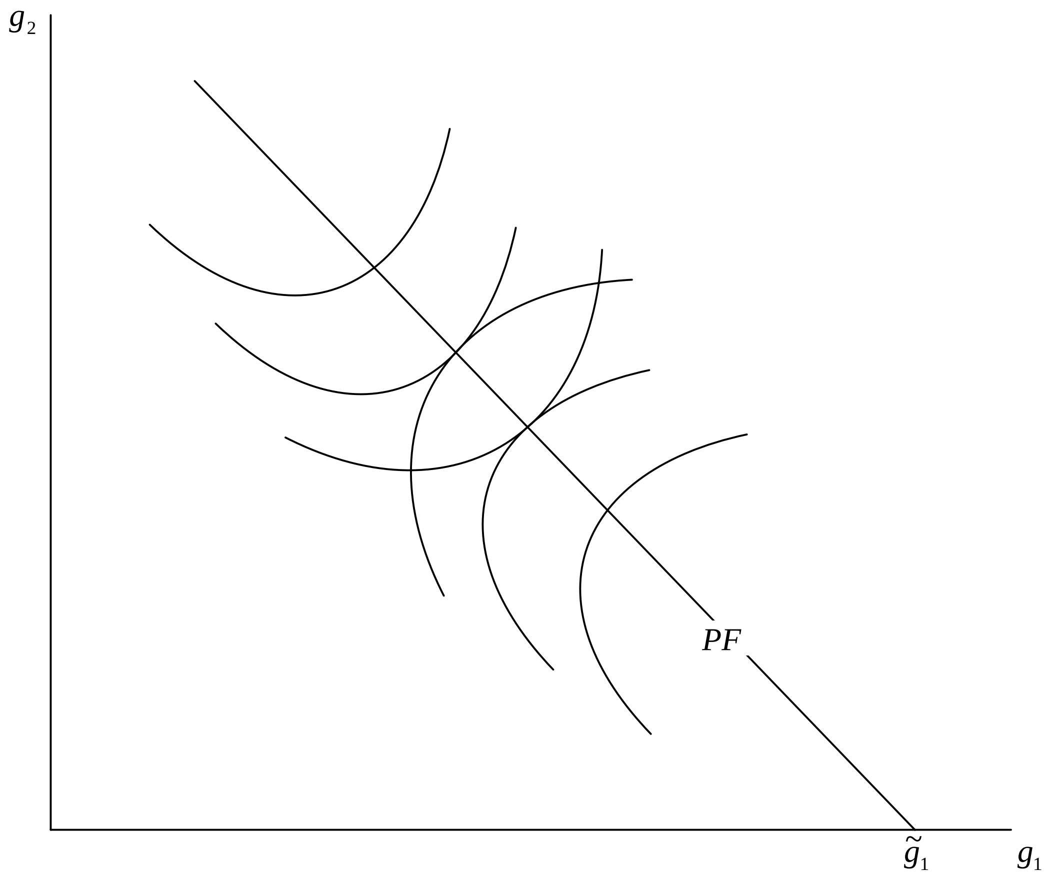

Figure 5-9

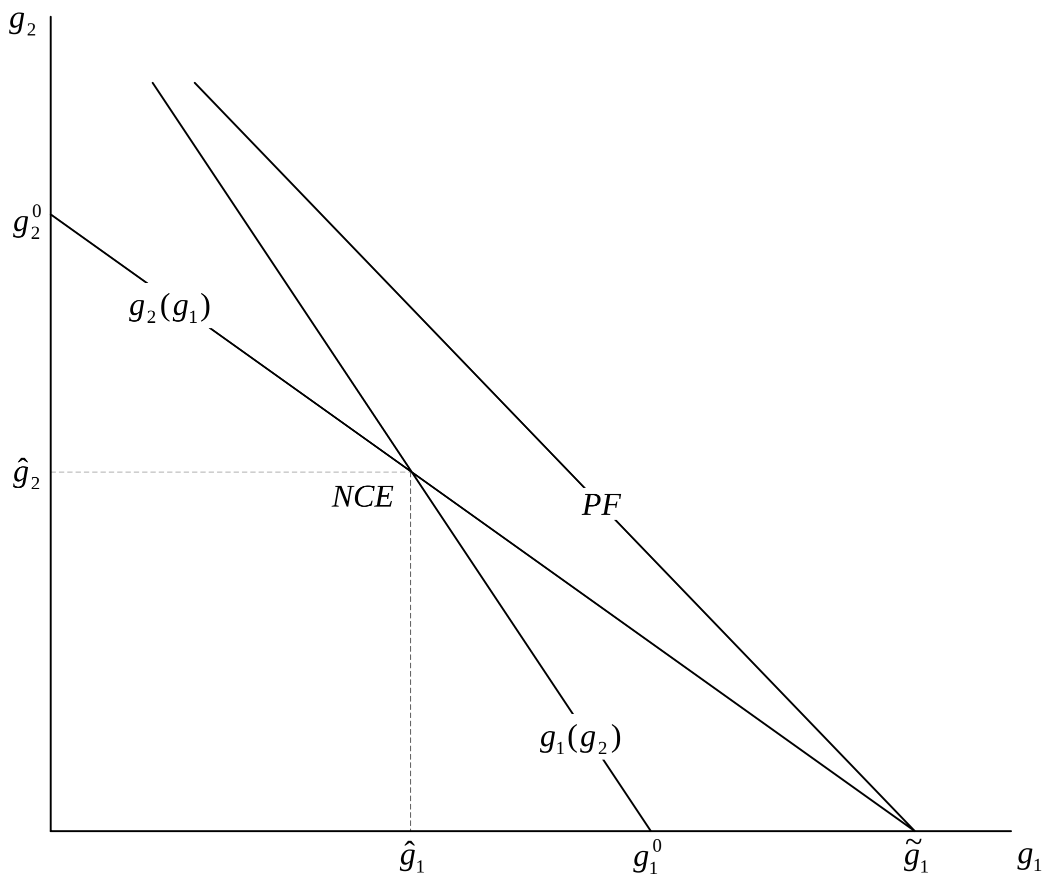

Figure 5-10

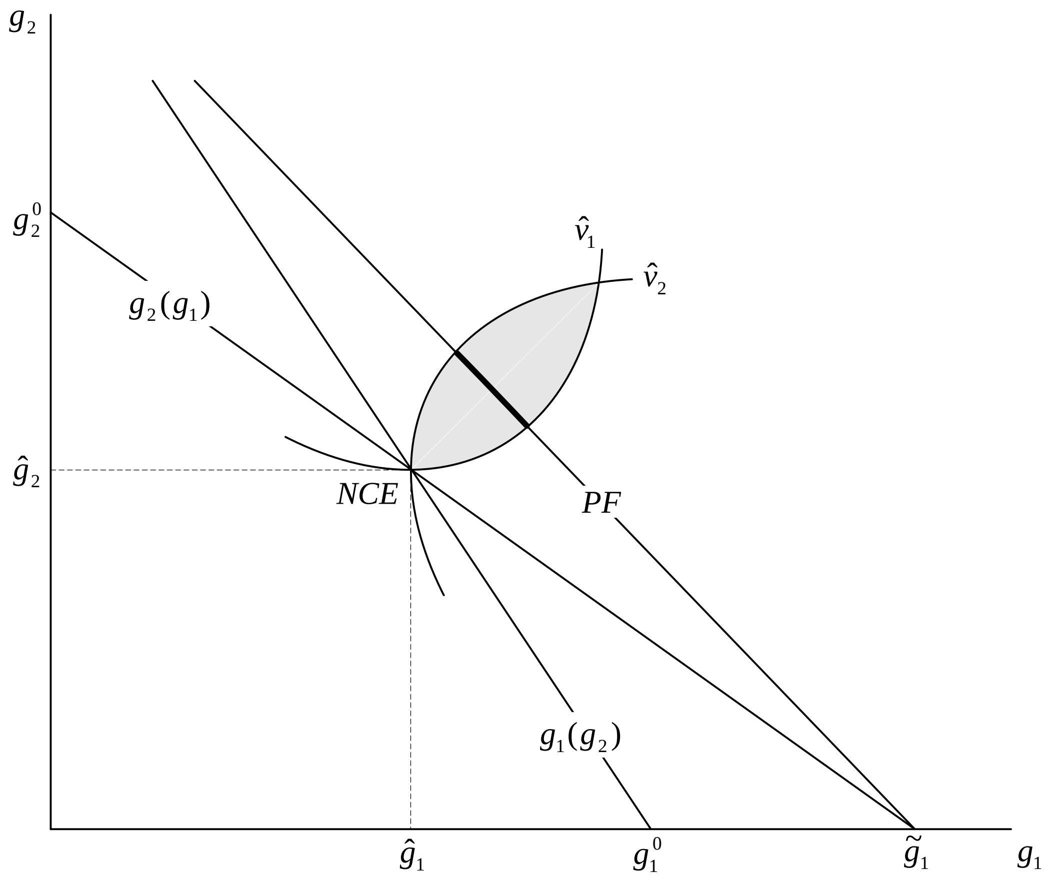
g_2
g_2^0
$g_2(g_1)$
$\hat{v}_1$
$\hat{v}_2$
$\hat{g}_2$
NCE
PF
$g_1(g_2)$
$\hat{g}_1$
g_1^0
$\tilde{g}_1$
g_1

Figure 5-11

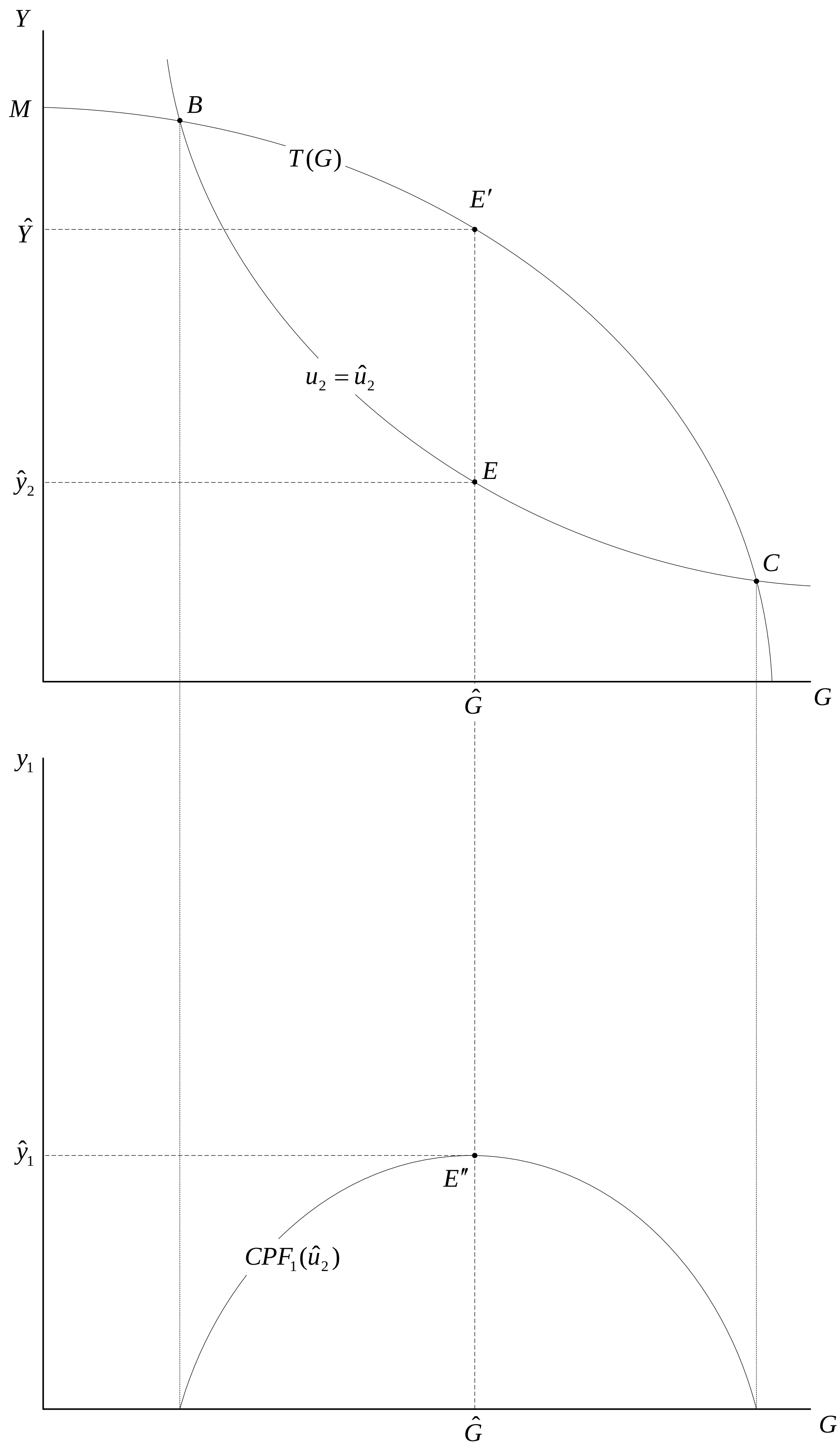

Figure 5-12

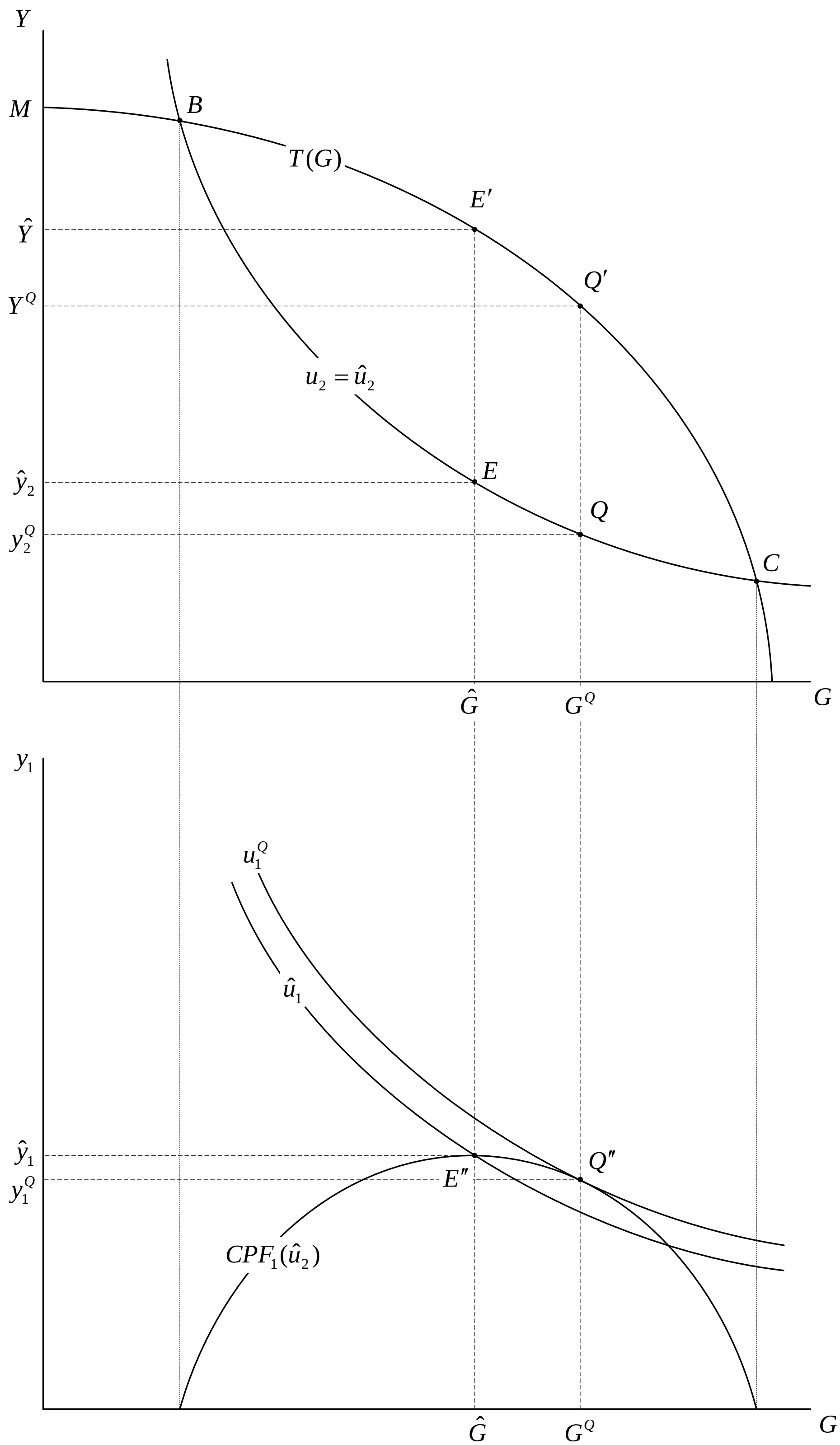
Y
M
B
T(G)
E′
Ŷ
Q′
Y^Q
u₂ = û₂
ŷ₂
E
Q
y₂^Q
C
Ĝ
G^Q
G
y₁
u₁^Q
û₁
ŷ₁
Q″
y₁^Q
E″
CPF₁(û₂)
Ĝ
G^Q
G

Figure 5-13

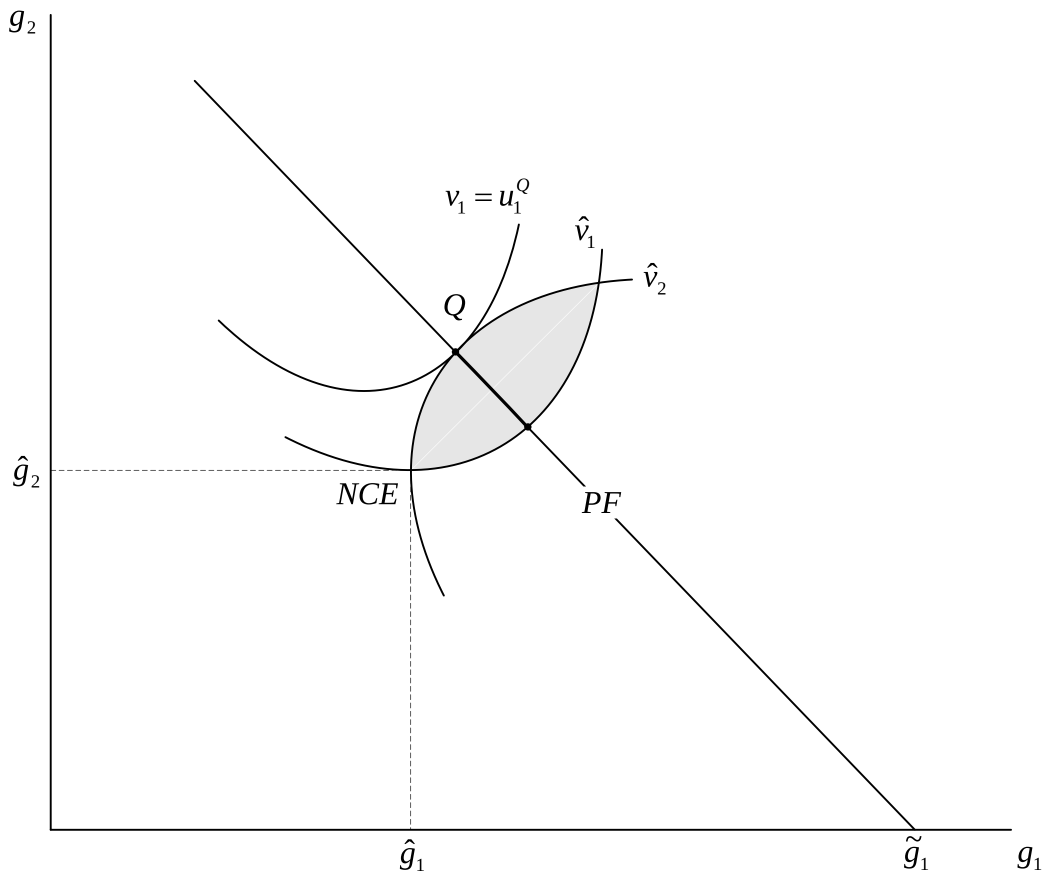

Figure 5-14

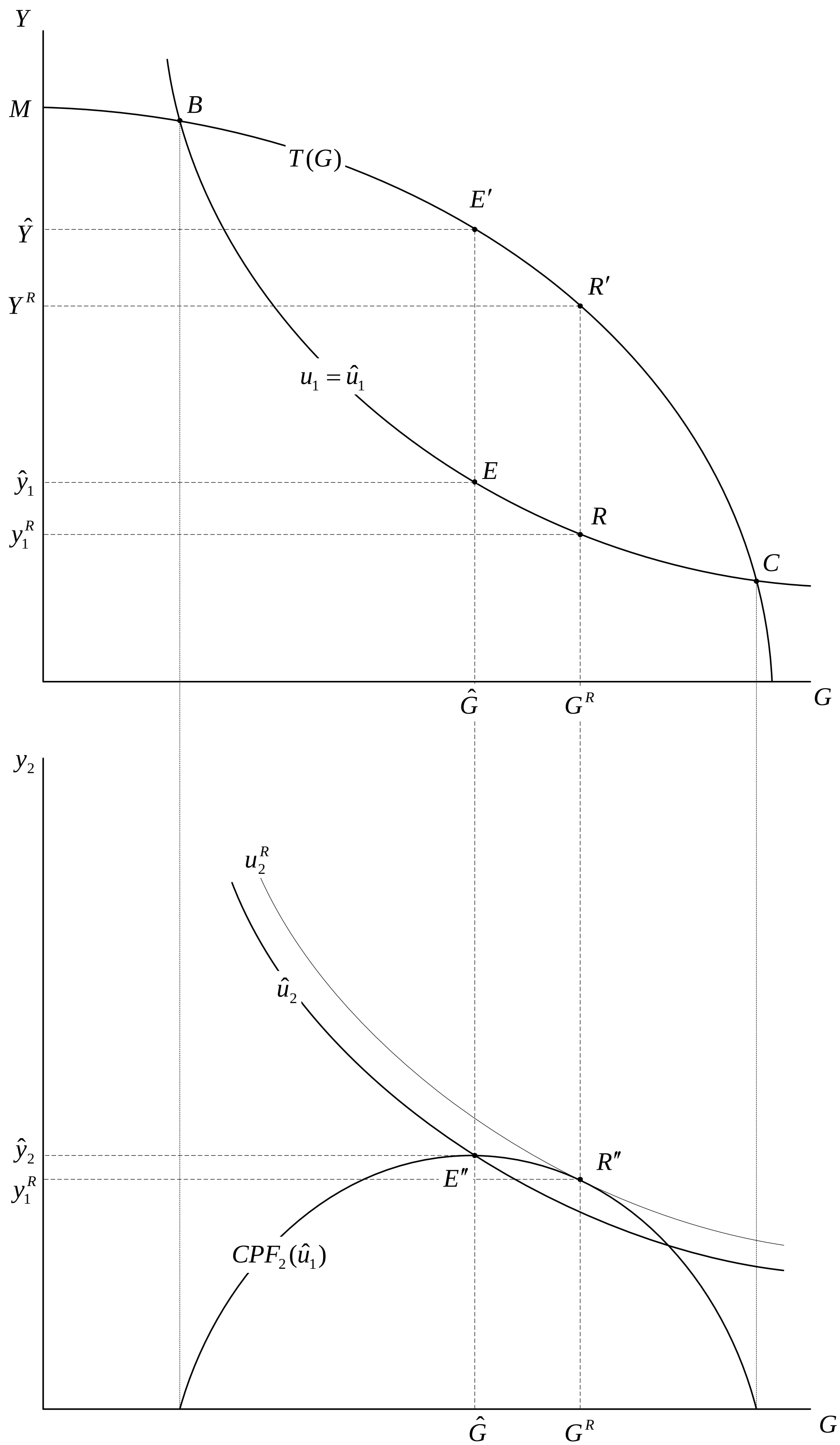
Y
M
B
T(G)
E′
Ŷ
Y^R
R′
u₁ = û₁
ŷ₁
E
y₁^R
R
C
Ĝ
G^R
G
y₂
u₂^R
û₂
ŷ₂
R″
y₁^R
E″
CPF₂(û₁)
Ĝ
G^R
G

Figure 5-15

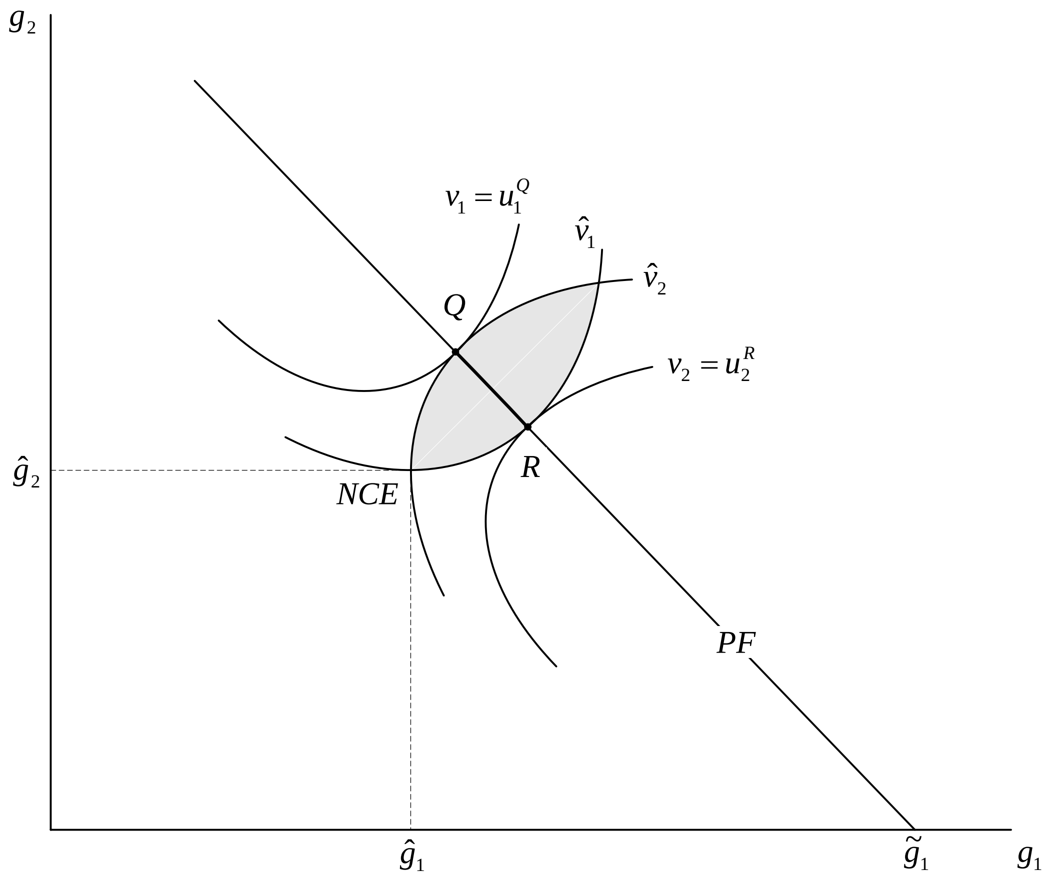

Figure 5-16