

6. CHOICE UNDER UNCERTAINTY

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6.1 INTRODUCTION

Many decisions are made in the presence of uncertainty. We will we characterize that uncertainty in terms of a set of possible states of nature with associated probabilities. Each possible state of nature is in turn associated with a value of wealth for the decision-maker. Decisions must be made prior to the resolution of the uncertainty (that is, prior to decision-maker knowing which state of nature will be realized).

We begin by constructing a payoff function expressed in terms of wealth and prices.

6.2 THE INDIRECT UTILITY FUNCTION

The indirect utility function is a payoff function constructed from the utility maximization problem that we examined in Topic 3.3-3, where an agent with wealth level m chose consumption levels for two goods (x and y) and leisure:

$$(6.1) \quad \max_{x, y, l} u_i(x, y, l) \quad \text{subject to} \quad p_x x + p_y y + wl = m$$

Recall that the maximizing solution involves two tangency conditions:

$$(6.2) \quad MRS_{xy} = \frac{p_x}{p_y}$$

and

$$(6.3) \quad MRS_{ly} = \frac{w}{p_y}$$

which can be solved in combination with the budget constraint to find optimal consumption values for x , y and l as functions of prices, and w and m . Let $x^*(p, w, m)$, $y^*(p, w, m)$ and $l^*(p, w, m)$ denote these optimal solutions, where p is the vector of prices.

What is the value of the utility function at this optimal solution?

If we substitute the optimal solutions back into the utility function we obtain

$$(6.4) \quad v(p, w, m) = u(x^*(p, w, m), y^*(p, w, m), l^*(p, w, m))$$

This is the **indirect utility function** (IUF). It measures the value of the objective function at the optimum, expressed as a function of the parameters in the budget constraint. (It is a particular type of a **maximum value function**).

Cobb-Douglas Example

Suppose preferences are Cobb-Douglas:

$$(6.5) \quad u(x, y, l) = x^\alpha y^\beta l^\delta$$

The utility-maximization conditions are

$$(6.6) \quad \frac{\alpha y}{\beta x} = \frac{p_x}{p_y}$$

$$(6.7) \quad \frac{\delta y}{\beta l} = \frac{w}{p_y}$$

$$(6.8) \quad p_x x + p_y y + w l = m$$

Recall that these equations can be solved by substitution. In particular, use (6.6) and (6.7) to express x and l in terms of y , and then substitute these into (6.8) and solve for y :

$$(6.9) \quad y(p_Y, m) = \frac{\beta m}{(\alpha + \beta + \delta)p_Y}$$

Now substitute this solution for y into (6.6) and solve for x , and into (6.7) and solve for l :

$$(6.10) \quad x(p_X, m) = \frac{\alpha m}{(\alpha + \beta + \delta)p_X}$$

$$(6.11) \quad l(w, m) = \frac{\delta m}{(\alpha + \beta + \delta)w}$$

Substitution of these optimal solutions into the objective function yields the IUF:

$$(6.12) \quad v(p, w, m) = \left(\frac{\alpha m}{(\alpha + \beta + \delta)p_X} \right)^\alpha \left(\frac{\beta m}{(\alpha + \beta + \delta)p_Y} \right)^\beta \left(\frac{\delta m}{(\alpha + \beta + \delta)w} \right)^\delta$$

where p is a vector of prices.

For our purposes, we are primarily interested in the relationship between v and m , so we will write this IUF as

$$(6.13) \quad v(p, w, m) = b(p, w)m^\theta$$

where

$$(6.14) \quad b(p, w) = \left(\frac{\alpha}{(\alpha + \beta + \delta)p_X} \right)^\alpha \left(\frac{\beta}{(\alpha + \beta + \delta)p_Y} \right)^\beta \left(\frac{\delta}{(\alpha + \beta + \delta)w} \right)^\delta$$

and

$$(6.15) \quad \theta = \alpha + \beta + \delta$$

For the most part we will assume that prices and the wage are fixed (focusing instead on uncertainty about m), so to simplify notation with will drop the functional dependence of v on p and w , and simply write $v(m)$ to denote the indirect utility function.¹

¹ The interpretation of $v(p, w, m)$ is actually somewhat more complicated than described here. In a more advanced course you would be introduced to the concept of a Von Neuman-Morgenstern utility function

What is the shape of $v(m)$ when plotted against m ? This depends on the size of θ .

Take the derivative of $v(m)$ with respect to m :

$$(6.16) \quad \frac{\partial v(m)}{\partial m} = \theta b(p, w) m^{\theta-1}$$

The sign of this derivative tells us the slope of $v(m)$ against m . It is positive: higher wealth leads to higher utility. The more interesting property is how that slope changes as m rises. That property is embodied in the second derivative of $v(m)$ with respect to m :

$$(6.18) \quad \frac{\partial^2 v(m)}{\partial m^2} = \theta(\theta - 1)b(p) m^{\theta-2}$$

The sign of this second derivative depends critically on the size of θ relative to one:

- if $\theta < 1$ then the second derivative is negative: $v(m)$ is increasing at a *decreasing* rate (as depicted in **Figure 6-1**). That is, $v(m)$ is strictly concave in m .
- if $\theta = 1$ then the second derivative is zero: $v(m)$ is increasing at a *constant* rate (as depicted in **Figure 6-2**). That is, $v(m)$ is linear in m .
- if $\theta > 1$ then the second derivative is positive: $v(m)$ is increasing at an *increasing* rate (as depicted in **Figure 6-3**). That is, $v(m)$ is strictly convex in m .

The IUF we have examine here is just an example, derived from Cobb-Douglas preferences. We can of course construct an IUF from any underlying preferences but in all cases one of its key properties – in the context of choice under uncertainty – is its curvature against m : is it strictly concave, linear or strictly convex? We will soon see why this is so important.

defined over wealth. However, for our purposes we can think of $v(p, w, m)$ in terms of the underlying utility function over goods despite that interpretation not being quite correct.

6.3 PROSPECTS AND THEIR PROPERTIES

Suppose an agent faces some uncertainty about her level of wealth. This uncertainty could arise from a variety of sources. For example: she may lose her job; her investment portfolio may perform badly; her house might burn down; she may become ill and be unable to work.

Let us characterize this uncertainty in terms of a prospect.

A **prospect** P is a set of state-contingent values m_i (which we will interpret here to be wealth values) and associated probabilities π_i that satisfy unitarity:

$$(6.19) \quad P = \{m_1, m_2, \dots, m_n; \pi_1, \pi_2, \dots, \pi_n\} \quad \text{such that} \quad \sum_{i=1}^n \pi_i = 1$$

We will sometimes refer to the n states that can possibly arise under the prospects as the “states of nature” associated with the prospect.

A **certain prospect** is one for which $m_i = m \forall i$. That is, wealth is the same in all states.

The **expected value** of a prospect is the expected value of the wealth outcomes under that prospect:

$$(6.20) \quad \mathbf{E}[P] = \mathbf{E}[m] = \sum_{i=1}^n \pi_i m_i$$

The **variance** of a prospect is the variance of the wealth outcomes under that prospect:

$$(6.21) \quad \sigma^2[P] = \sigma^2[m] = \sum_{i=1}^n \pi_i (m_i - \mu)^2$$

where $\mu \equiv \mathbf{E}[m]$.

Example 1

Consider a prospect with two possible states:

$$P_A = \{100, 64; \frac{1}{2}, \frac{1}{2}\}$$

The expected value of this prospect is

$$\mathbf{E}[P_A] = \frac{1}{2}(100) + \frac{1}{2}(64) = 82$$

The variance of this prospect is

$$\sigma^2[P_A] = \frac{1}{2}(100 - 82)^2 + \frac{1}{2}(64 - 82)^2 = 324$$

Example 2

Consider a prospect with three possible states:

$$P_B = \{196, 64, 36; \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$$

The expected value of this prospect is

$$\mathbf{E}[P_B] = \frac{1}{4}(196) + \frac{1}{2}(64) + \frac{1}{4}(36) = 90$$

The variance of this prospect is

$$\sigma^2[P_B] = \frac{1}{4}(196 - 90)^2 + \frac{1}{2}(64 - 90)^2 + \frac{1}{4}(36 - 90)^2 = 3876$$

Note that this prospect has a higher expected value than P_A from example 1 but it has a much higher variance. We will say that P_B has higher risk than P_A .

Example 3

Consider a certain prospect with two possible states:

$$P_C = \{82, 82; \frac{1}{2}, \frac{1}{2}\}$$

The expected value of this prospect is

$$\mathbf{E}[P_C] = \frac{1}{2}(82) + \frac{1}{2}(82) = 82$$

The variance of this prospect is

$$\sigma^2[P_C] = \frac{1}{2}(82 - 82)^2 + \frac{1}{2}(82 - 82)^2 = 0$$

This prospect has no risk at all.

Risk vs. Reward

Prospects are sometimes compared on the basis of their “risk-to-reward ratio”, defined as

$$(6.22) \quad \rho = \frac{\sigma}{\mu}$$

where σ is the standard deviation of the prospect, and μ is its expected value.

For the three examples above: $\rho_A = 0.2195$; $\rho_B = 0.6917$; and $\rho_C = 0$. Note that prospect B has a higher expected value than prospect A , but the latter has a lower risk-to-reward ratio.

6.4 EXPECTED UTILITY THEORY

A more general approach to comparing prospects is to assess their expected utility from the perspective of the agent facing those prospects.

The expected utility of a prospect is the probability-weighted sum of the IUF values associated with each possible realization of the prospect:

$$(6.23) \quad \mathbf{E}[v(m)] = \sum_{i=1}^n \pi_i v(m_i)$$

Expected utility theory asserts that when people can choose among alternative prospects, they will choose the prospect that yields the highest expected utility. That is, economic agents are expected-utility-maximizers.

Example

Suppose an agent has the IUF from section 6.2:

$$v(m) = b(p)m^\theta$$

and suppose he can choose between prospects P_A and P_C from section 6.3:

$$P_A = \{100, 64; \frac{1}{2}, \frac{1}{2}\}$$

$$P_C = \{82, 82; \frac{1}{2}, \frac{1}{2}\}$$

His expected utility from P_A is

$$\mathbf{E}[v(m_A)] = \frac{1}{2}b(p)(100)^\theta + \frac{1}{2}b(p)(64)^\theta = b(p)\left(\frac{1}{2}(100)^\theta + \frac{1}{2}(64)^\theta\right)$$

Note that $b(p)$ can be taken outside the brackets because this term is the same in all states of nature (because prices are not subject to uncertainty in this prospect).

His expected utility from P_C is

$$\mathbf{E}[v(m_C)] = \frac{1}{2}b(p)(82)^\theta + \frac{1}{2}b(p)(82)^\theta = b(p)(82)^\theta$$

Again note that $b(p)$ can be taken outside the brackets because this term is the same in all states of nature. This in turn means that the ranking of the two prospects (which differ only in their wealth outcomes) does not depend on $b(p)$ at all.

This irrelevance of prices in the ranking of prospects over wealth is not a general result but it holds for a class of utility functions called homothetic functions.

Homothetic Utility Functions

We will not pursue the details here, but the key property of homothetic utility functions is that they give rise to demand functions that are linear in income, like those in (6.10) – (6.12) above. The Cobb-Douglas function is a member of this homothetic function family.

For pedagogical purposes it is helpful to confine consideration to the homothetic-family case, otherwise the mathematics can become so complicated that it can distract from the key economic ideas of interest. Moreover, if we restrict our analysis in this way then we can make a helpful normalization that imposes no further restrictions on our results but simplifies how we get to those results. In particular, we can set $b(p) = 1$ by implicitly choosing a numeraire price that effectively forces the term in (6.15) to be exactly one.

This means that if we work with homothetic utility functions, then we can effectively specify the IUF as

$$(6.24) \quad v(m) = m^\theta$$

and still capture most of the important aspects of choice under uncertainty provided that uncertainty relates to wealth and not prices. With this assumption in place, let us revisit the example and consider the choice between prospects.

Example Revisited

Recall the two prospects:

$$P_A = \{100, 64; \frac{1}{2}, \frac{1}{2}\}$$

$$P_C = \{82, 82; \frac{1}{2}, \frac{1}{2}\}$$

These two prospects have the same expected value:

$$\mathbf{E}[P_A] = \frac{1}{2}(100) + \frac{1}{2}(64) = 82$$

$$\mathbf{E}[P_C] = \frac{1}{2}(82) + \frac{1}{2}(82) = 82$$

but P_A has a higher variance.

Using the IUF from (6.24), the agent's expected utility from P_A is

$$\mathbf{E}[v(m_A)] = \frac{1}{2}(100)^\theta + \frac{1}{2}(64)^\theta$$

and his expected utility from P_C is

$$\mathbf{E}[v(m_C)] = (82)^\theta$$

Comparing these expected utilities in terms of θ reveals the following:

- if $\theta < 1$ then $\mathbf{E}[v(m_A)] < \mathbf{E}[v(m_C)]$
- if $\theta = 1$ then $\mathbf{E}[v(m_A)] = \mathbf{E}[v(m_C)]$
- if $\theta > 1$ then $\mathbf{E}[v(m_A)] > \mathbf{E}[v(m_C)]$

6.5 ATTITUDES TOWARDS RISK

Why is the size of θ so important here? Recall from section 6.2 that the size of θ relative to one determines the curvature of the IUF. This curvature embodies the attitude of this agent toward risk:

- if $v(m)$ is strictly concave in m then the agent is **risk averse**: if two prospects have the same expected value, he prefers the one with the lowest variance.
- if $v(m)$ is linear in m then the agent is **risk neutral**: if two prospects have the same expected value, he is indifferent between them regardless of their relative variance.
- if $v(m)$ is strictly convex in m then the agent is **risk loving**: if two prospects have the same expected value, he prefers the one with the highest variance.

Let us calculate the expected utility of prospects P_A and P_C for three different values of θ , corresponding to the three possible attitudes towards risk.

Risk Averse

Suppose $\theta = \frac{1}{2}$, then

$$\mathbf{E}[v(m_A)] = \frac{1}{2}(100)^{\frac{1}{2}} + \frac{1}{2}(64)^{\frac{1}{2}} = 9 \quad \text{and} \quad \mathbf{E}[v(m_C)] = (82)^{\frac{1}{2}} = 9.055$$

Thus, the agent prefers P_C to P_A . This case is illustrated in **Figure 6-4**.

Risk Neutral

Suppose $\theta = 1$, then

$$\mathbf{E}[v(m_A)] = \frac{1}{2}(100)^1 + \frac{1}{2}(64)^1 = 82 \quad \text{and} \quad \mathbf{E}[v(m_C)] = (82)^1 = 82$$

Thus, the agent is indifferent between P_C and P_A . This case is illustrated in **Figure 6-5**.

Risk Loving

Suppose $\theta = \frac{3}{2}$, then

$$\mathbf{E}[v(m_A)] = \frac{1}{2}(100)^{\frac{3}{2}} + \frac{1}{2}(64)^{\frac{3}{2}} = 756 \quad \text{and} \quad \mathbf{E}[v(m_C)] = (82)^{\frac{3}{2}} = 742.45$$

Thus, the agent prefers P_A to P_C . This case is illustrated in **Figure 6-6**.

In each of these three cases, we are comparing two prospects that have the same expected value, and the attitude towards risk determines which one is preferred. This leads us to the following more general characterization of risk attitudes.

For a **risk averse** agent:

$$\mathbf{E}[v(m)] < v(\mathbf{E}[m])$$

That is, for a risk-averse agent, the expected utility of a prospect is less than the utility of the expected value of that prospect.

For a **risk neutral** agent:

$$\mathbf{E}[v(m)] = v(\mathbf{E}[m])$$

That is, for a risk-neutral agent, the expected utility of a prospect is equal to the utility of the expected value of that prospect.

For a **risk loving** agent:

$$\mathbf{E}[v(m)] > v(\mathbf{E}[m])$$

That is, for a risk-loving agent, the expected utility of a prospect is greater than the utility of the expected value of that prospect.

Empirical evidence indicates that most people are risk averse. (Some people have a penchant for high-risk activities like high-speed motorcycling but typically it is the adrenaline rush they enjoy, not the risk *per se*). We will henceforth focus primarily on the risk-averse case.

6.6 CERTAINTY-EQUIVALENT WEALTH AND THE RISK PREMIUM

Suppose an agent faces the uncertain prospect

$$P_A = \{100, 64; \frac{1}{2}, \frac{1}{2}\}$$

but is offered a certain prospect

$$P_D = \{m, m; \pi, 1 - \pi\}$$

instead. What value of m would make this agent just indifferent between P_A and P_D ?

The value of m that makes the agent indifferent between P_A and P_D is called the **certainty-equivalent wealth** associated with prospect P_A .

Example

Consider an example where the agent has IUF given by $v(m) = m^{\frac{1}{2}}$.

We know that the expected utility from P_A is

$$\mathbf{E}[v(m_A)] = \frac{1}{2}(100)^{\frac{1}{2}} + \frac{1}{2}(64)^{\frac{1}{2}} = 9$$

In comparison, the expected utility from the certain prospect is

$$\mathbf{E}[v(m_D)] = v(m_D) = m^{\frac{1}{2}}$$

The certainty-equivalent wealth is \hat{m} such that $\mathbf{E}[v(m_D)] = \mathbf{E}[v(m_A)]$. Thus, \hat{m} is the solution to

$$\hat{m}^{\frac{1}{2}} = 9$$

That is, $\hat{m} = 81$.

Note the certainty-equivalent wealth associated with P_A is less than the expected value of P_A , which is

$$\mathbf{E}[m] = \frac{1}{2}(100) + \frac{1}{2}(64) = 82$$

The difference between these two values is called the **risk premium** associated with prospect, denoted R . That is,

$$R = \mathbf{E}[m] - \hat{m}$$

In the example, $R = 82 - 81 = 1$.

In general, for the agent to be indifferent between an uncertain prospect and a certain prospect with value \hat{m} , the uncertain prospect must have an associated expected value of $\hat{m} + R$, where R is the premium required to compensate for the risk.

The size and sign of R for any given prospect naturally depends on the risk attitude of the agent facing that prospect. In particular, if the agent is

- risk averse, then $R > 0$
- risk neutral, then $R = 0$
- risk loving, then $R < 0$

Figure 6-7 illustrates the risk premium for a prospect $P = \{m_1, m_2; \frac{1}{2}, \frac{1}{2}\}$ for a risk-averse agent. Note that we can usefully identify $\mathbf{E}[v(m)]$ as the value on a chord below the IUF evaluated at $\mathbf{E}[m]$. Why? This linear segment tells us how a risk-neutral agent would view this prospect, and for a risk-neutral agent we know that $\mathbf{E}[v(m)] = v(\mathbf{E}[m])$

6.7 AN APPLICATION: THE DEMAND FOR INSURANCE

Suppose an agent has current wealth m but with probability π she will suffer a loss L . She therefore faces an uncertain prospect:

$$P = \{m, m - L; 1 - \pi, \pi\}$$

Suppose further that she can buy insurance against loss at price r per dollar of coverage c . How much insurance will she buy?

We can think of her choice problem as one of choosing one prospect from a schedule of prospects:

$$P(c) = \{m - rc, m - rc - L + c; 1 - \pi, \pi\}$$

and she makes that choice to maximize her expected utility:

$$\max_c \pi v(m - rc - L + c) + (1 - \pi)v(m - rc)$$

Example

Suppose the agent has IUF given by $v(m) = m^{\frac{1}{2}}$. She has current wealth \$6300 but faces a loss of \$6000 with probability $\frac{1}{4}$. She can purchase insurance against that loss at a price of $r = \frac{1}{2}$ per dollar of coverage. Then her choice problem is

$$\max_c \frac{1}{4}(6300 - \frac{1}{2}c - 6000 + c)^{\frac{1}{2}} + \frac{3}{4}(6300 - \frac{1}{2}c)^{\frac{1}{2}}$$

Differentiate with respect to c and set this equal to zero:

$$\frac{1}{8}(6300 - \frac{1}{2}c - 6000 + c)^{-\frac{1}{2}}(-\frac{1}{2} + 1) + \frac{3}{8}(6300 - \frac{1}{2}c)^{-\frac{1}{2}}(-\frac{1}{2}) = 0$$

Multiply both sides by 16 and rearrange the equation to obtain

$$(6300 - \frac{1}{2}c - 6000 + c)^{\frac{1}{2}} = 3(6300 - \frac{1}{2}c)^{\frac{1}{2}}$$

We can then square both sides and solve the resulting equation for

$$c^* = 720$$

That is, the agent buys \$720 of coverage. Note this is far less than the size of the potential loss she faces; she is heavily under-insured.

Now let us calculate the expected profit made by the insurance company from this policy:

$$\begin{aligned} \mathbf{E}[\text{profit}] &= \pi(rc - c) + (1 - \pi)rc = (r - \pi)c \\ &= (\frac{1}{2} - \frac{1}{4})720 \\ &= 180 \end{aligned}$$

Suppose this insurance market is now opened up to competition, and companies compete for the agent's business until expected profit is driven to zero.

The zero-profit insurance price is $r = \pi$. This is called the **actuarially-fair** price because it accurately reflects the probability of a payout.

How much insurance will the agent buy if she is charged the actuarially-fair price?

At price $r = \frac{1}{4}$ her choice problem is now

$$\max_c \frac{1}{4} (6300 - \frac{1}{4}c - 6000 + c)^{\frac{1}{2}} + \frac{3}{4} (6300 - \frac{1}{4}c)^{\frac{1}{2}}$$

Solving this problem using the same procedure we used above yields

$$c^* = 6000$$

That is, she buys **full insurance**: her coverage will exactly cover her loss.

The Relationship Between the Risk Premium and the Demand for Insurance

Suppose a person is offered full insurance against a loss L at price r per dollar of coverage. Her total premium will be $T = rL$.

What is the maximum total premium she would be willing to pay for full insurance, denoted \hat{T} ?

We can calculate this as the solution to an indifference equation, in the same way we did to calculate the certainty-equivalent wealth.

Her expected utility without insurance is

$$\mathbf{E}[v^0] = \pi v(m - L) + (1 - \pi)v(m)$$

and her expected utility with full insurance is

$$\mathbf{E}[v^F] = \pi v(m - T) + (1 - \pi)v(m - T) = v(m - T)$$

We can find her maximum WTP for full insurance as the solution to $\mathbf{E}[v^0] = \mathbf{E}[v^F]$:

$$\pi v(m - L) + (1 - \pi)v(m) = v(m - \hat{T})$$

This is precisely the same indifference equation that defines the certainty-equivalent wealth. That is, $m - \hat{T}$ is the certainty-equivalent wealth associated with the uninsured prospect.

Moreover, we know that the risk premium for a prospect is by definition equal to the difference between expected wealth and the certainty-equivalent wealth. Thus, in this setting

$$R = (\pi(m - L) + (1 - \pi)m) - (m - \hat{T})$$

and this reduces to

$$R = \hat{T} - \pi L$$

Thus,

$$\hat{T} = \pi L + R$$

That is, the maximum total premium this person is willing to pay for full insurance is the expected loss plus the risk premium. For a risk neutral person, the risk premium is zero.

Implications of Asymmetric Information

Note that we have assumed here that the agent cannot influence π or L by taking a precautionary action, and that the insurance company can observe π and set an actuarially-fair price accordingly.

In practice, neither of these simplifying assumptions are likely to be satisfied. When we relax those assumptions, we introduce the possibility of moral hazard and adverse selection. These relate to asymmetric information about actions and characteristics respectively. They are the subject of our next topic.

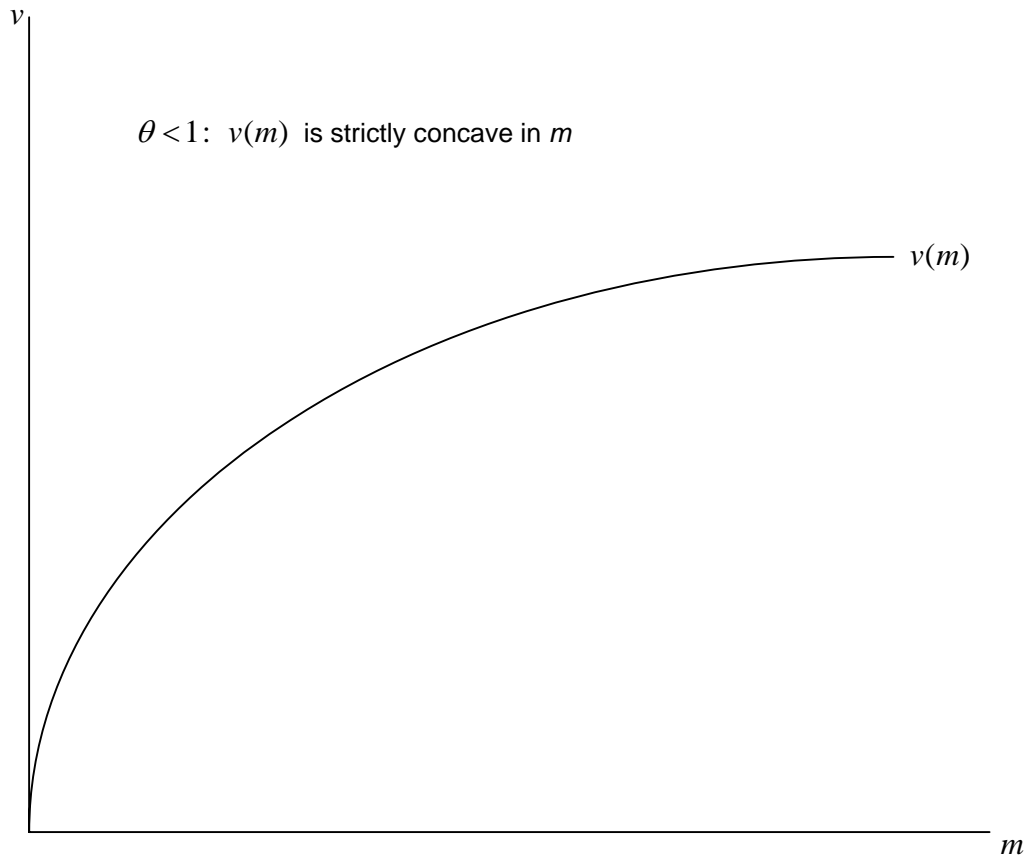


Figure 6-1

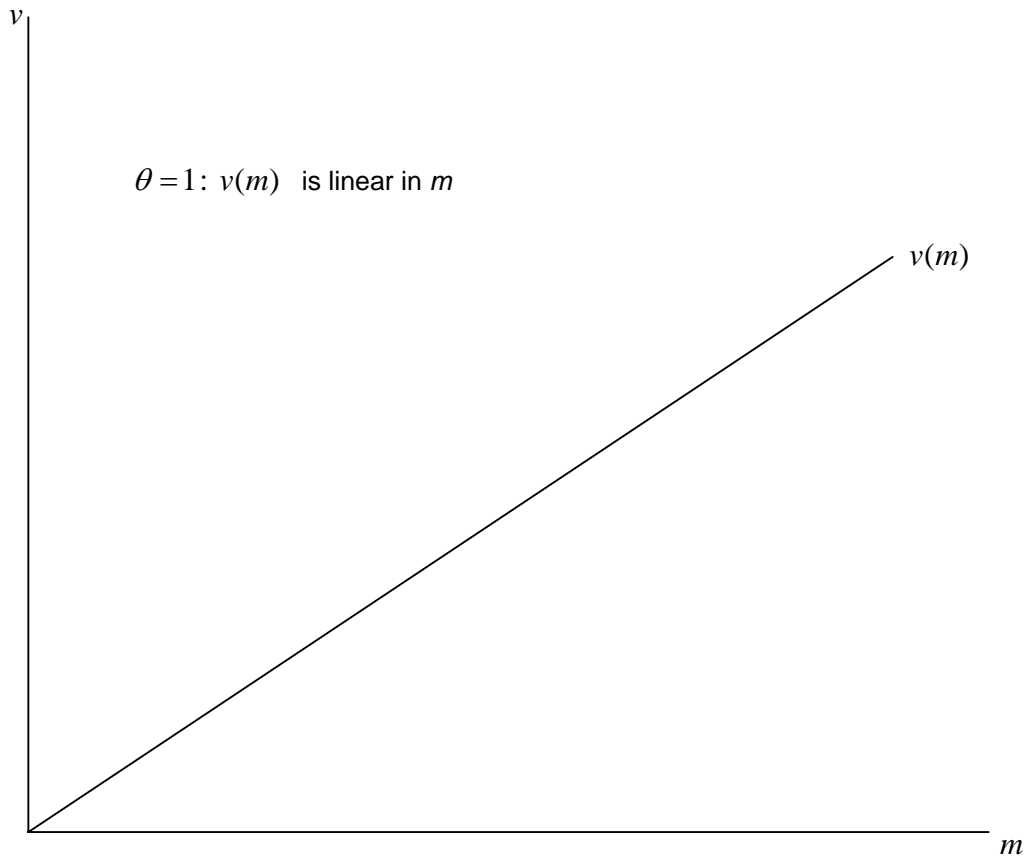


Figure 6-2

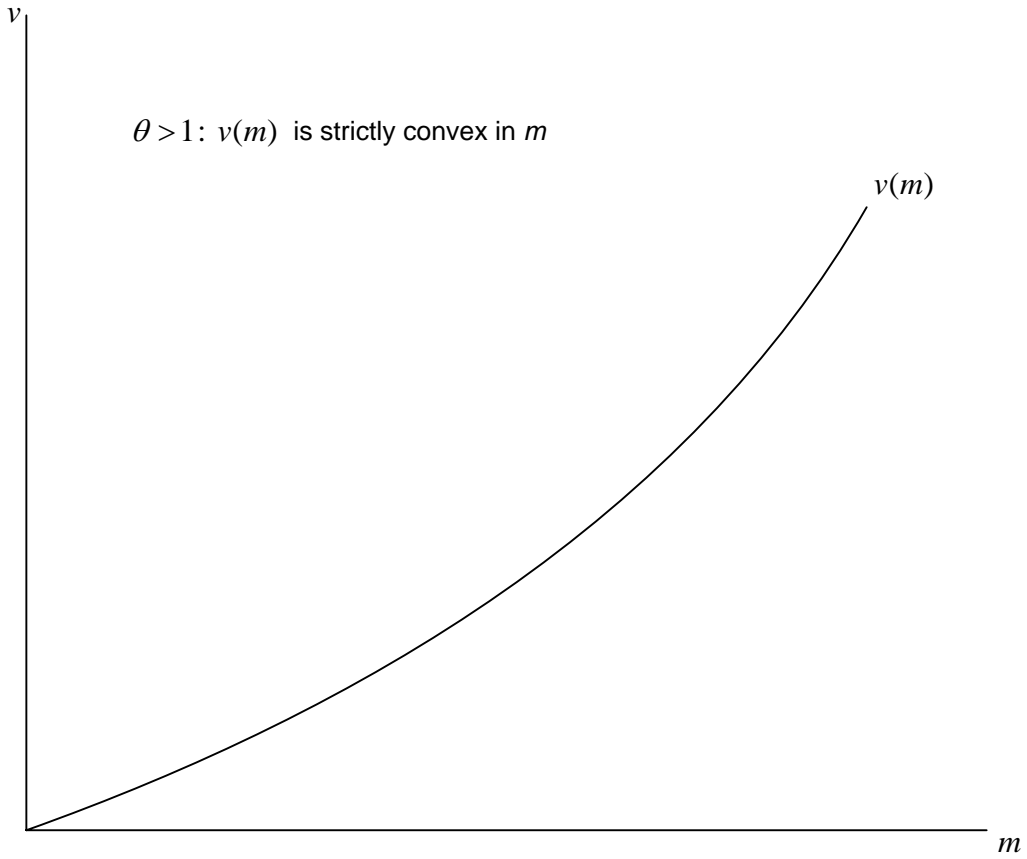


Figure 6-3

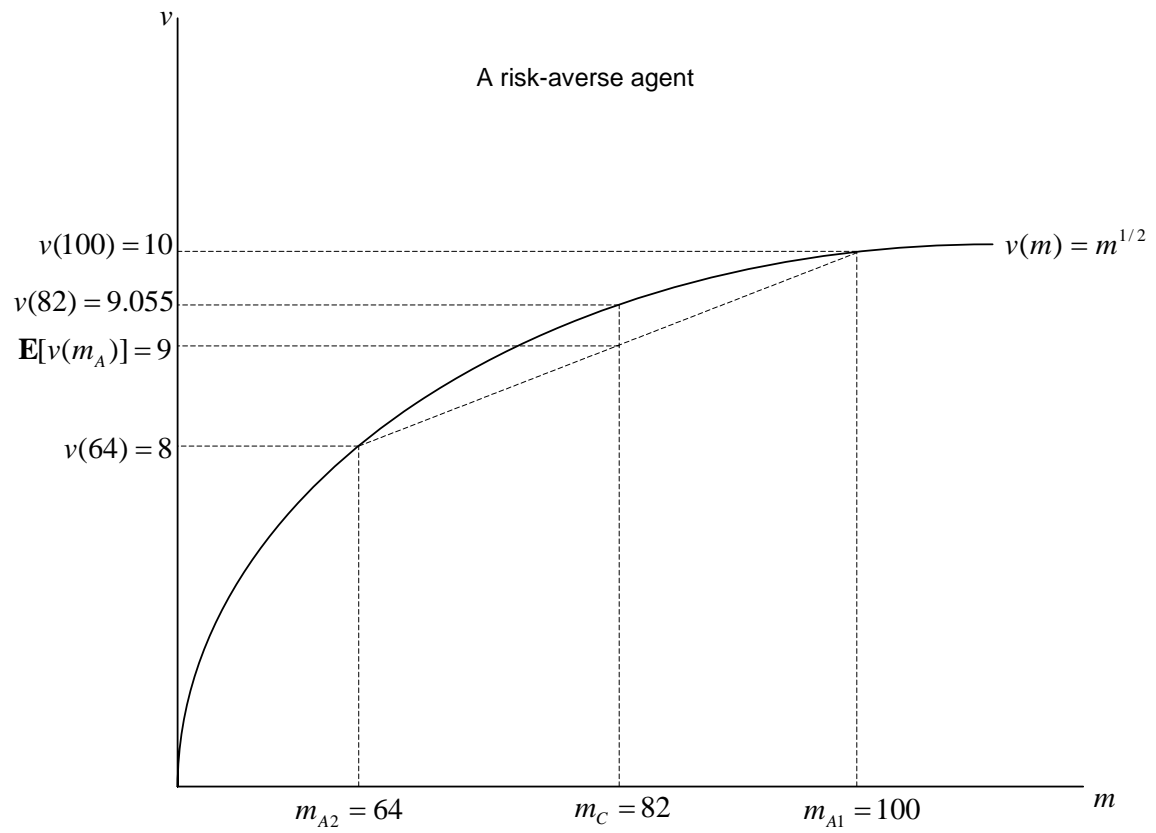


Figure 6-4

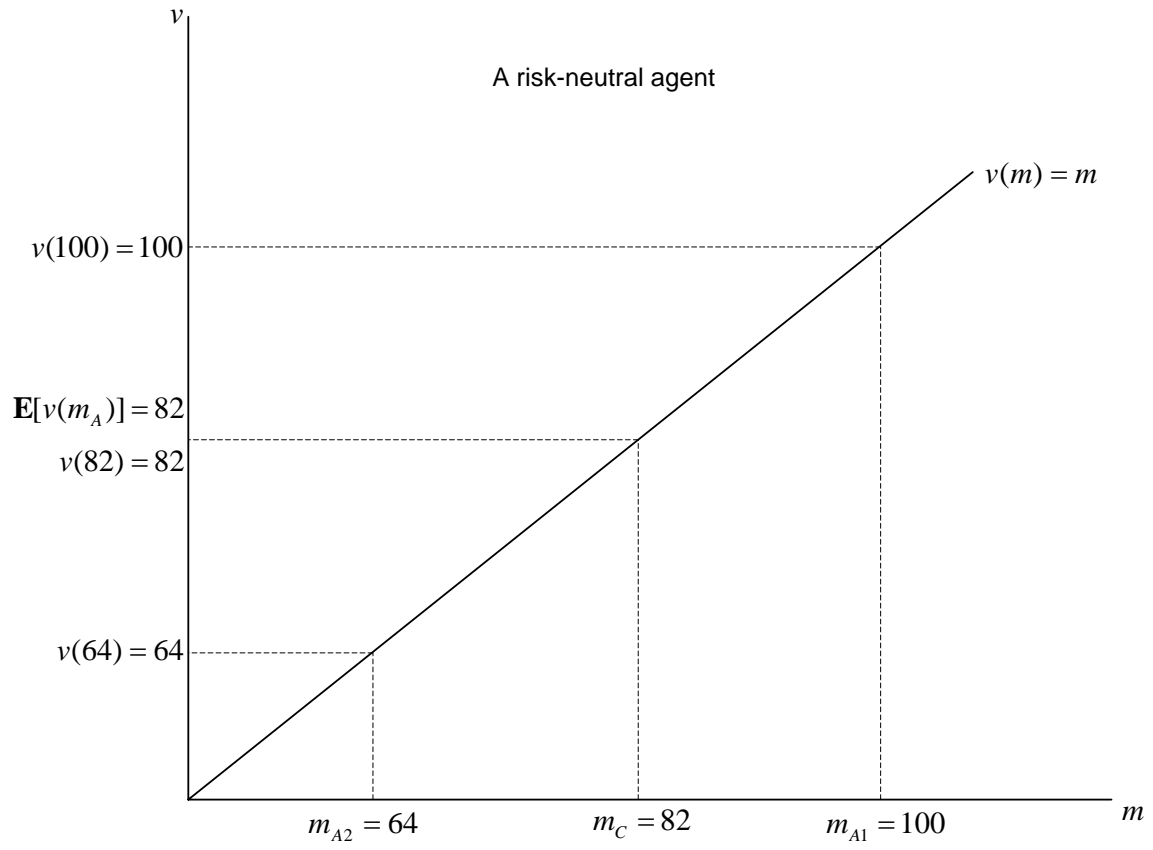


Figure 6-5

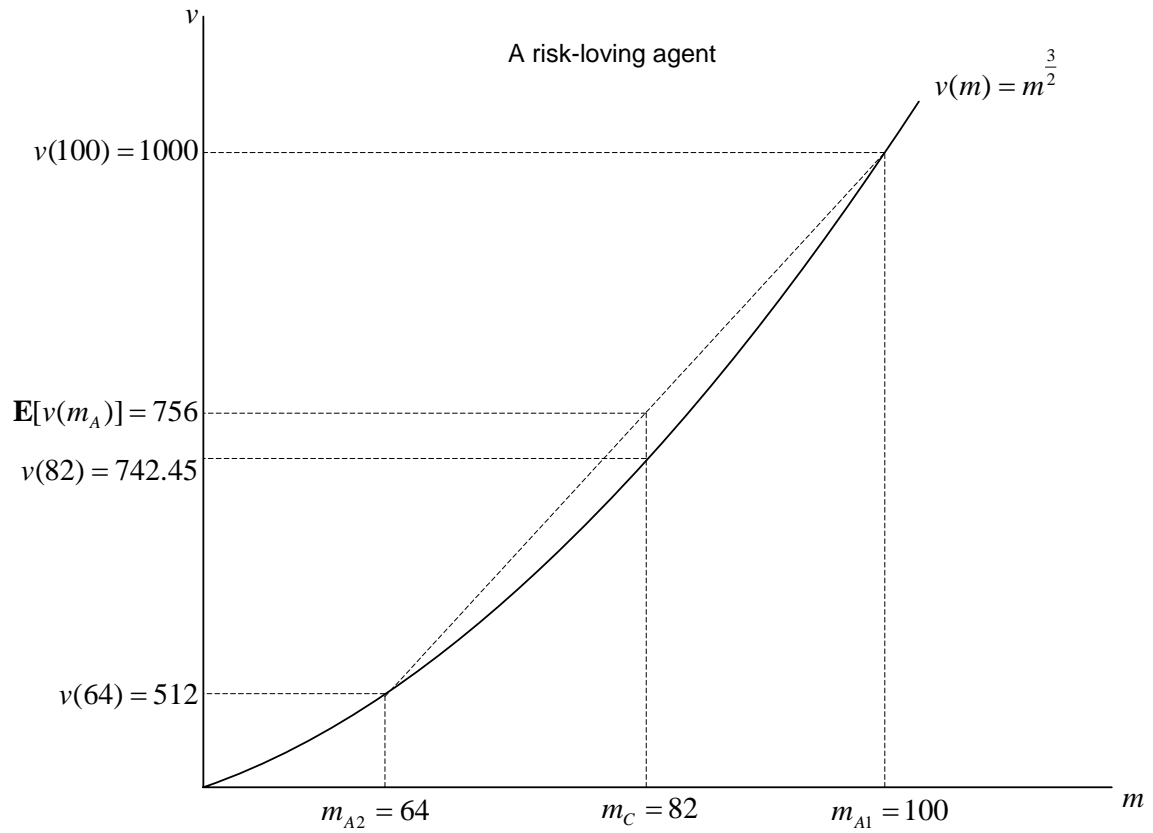


Figure 6-6

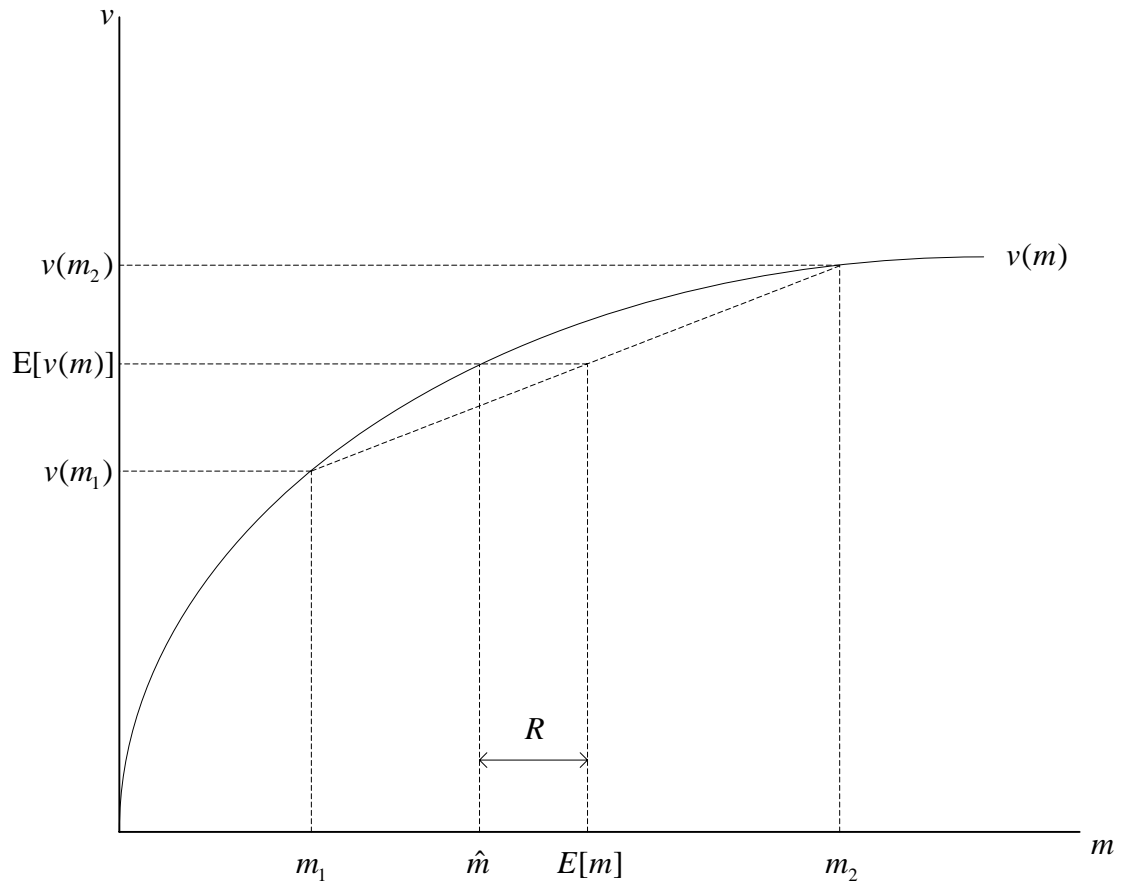


Figure 6-7