ECONOMICS 531
PROJECT 2

Abatement Cost for a Price-Taking Firm: a Simple Model

Consider a firm that produces output *y* using a production process $x \in [0,1]$ that generates emissions equal to

(1) $$e(y,x) = \psi(1-x)y$$

where $(1-x)$ is the *emissions-intensity* of production, which is a choice variable for the firm, and ψ is a technology parameter that for we will hold fixed (but could be endogenized as a technology choice). The firm is a price-taker in its output market, and its output price is *p*.

The variable cost of production is

(2) $$c(y,x) = \omega(1+\zeta x^2)y^2$$

where $\omega > 0$ and $\zeta > 0$. This cost function can be derived from a production process in which resources must be diverted away from production and into pollution control. (See the **Appendix** for details). This diversion of resources is costly, and so cleaner production is more costly than dirty production. Moreover, a cleaner production process raises the *marginal* cost of production. In particular, the marginal cost of production is

(3) $$m(y,x) \equiv \frac{\partial c}{\partial y} = 2\omega(1+\zeta x^2)y$$

and this function is increasing and strictly convex in *x*:

(4) $$\frac{\partial m}{\partial x} = 4\omega\zeta xy > 0 \quad \text{and} \quad \frac{\partial^2 m}{\partial x^2} = 4\omega\zeta y > 0$$

We will see that these properties of the production function will yield a "low-hanging fruit" element to the abatement process for this firm.

Your task is to code-up this model in Maple and examine its properties by following the five steps described below.

1. The Unregulated Private Optimum

The firm's unregulated profit-maximization problem is simply

(5) $$\max_{x,y} \; py - c(y,x)$$

Show that if the firm faces no regulatory constraints on its emissions then its profit is

(6) $$\hat{\pi}(p) = \frac{p^2}{4\omega}$$

and its emissions are

(7) $$\hat{e}(p) = \frac{p\psi}{2\omega}$$

2. The Impact of an Emissions Constraint

Now suppose the firm faces a regulatory constraint on its emissions: $e(y,x) \leq e$.

What is the critical value of *e* (expressed as a function of *p* and ψ) below which the emissions constraint is binding? Henceforth assume that the constraint is binding.

The firm's constrained profit-maximization problem is

(8) $$\max_{x,y} \; py - c(y,x) \quad \text{subject to} \quad \psi(1-x)y \leq e$$

Show that if the emissions constraint is binding, then the its supply function is

(9) $$\tilde{y}(p,e) = \frac{p\psi + 2\omega\zeta e}{2\omega(1+\zeta)\psi}$$

and its associated choice of production process is

(10) $$\tilde{x}(p,e) = 1 - \frac{e}{\tilde{y}(pe,)\psi} = \frac{p\psi - 2\omega e}{p\psi + 2\omega\zeta e}$$

Find profit at the constrained optimum, and show that it can be expressed as

(11) $$\tilde{\pi}(p,e) = \hat{\pi}(p) - \frac{(\hat{e}-e)^2}{2\gamma}$$

where $\hat{\pi}(p)$ is unregulated profit, $\hat{e}(p)$ is the unregulated level of emissions, and

(12) $$\gamma \equiv \frac{(1+\zeta)\psi^2}{2\zeta\omega}$$

is a summary parameter that we will use throughout.

Note that expressing profit is this way makes it clear that any regulatory requirement to reduce emissions below the unregulated private optimum will cause profit to fall below its unregulated value, $\hat{\pi}(p)$. Construct a plot like **Figure 1** to illustrate this relationship. (Label this plot "Figure 1" in your output).

3. Abatement Cost

Abatement cost for the firm is the loss of profit associated with meeting the emissions constraint. Thus, from (11) we have

(13) $$AC(e) \equiv \hat{\pi}(p) - \tilde{\pi}(p,e) = \frac{(\hat{e}-e)^2}{2\gamma}$$

We can now derive marginal abatement cost for the firm as

(14) $$MAC(e) = -\frac{\partial AC(e)}{\partial e} = \frac{\hat{e}-e}{\gamma}$$

Note that this is the same marginal abatement cost function from our example in Topic 3.6 from the slides.

Note too that we can derive marginal abatement cost directly as

(15) $$MAC(e) = \frac{\partial \tilde{\pi}}{\partial e} = \frac{\hat{e}-e}{\gamma}$$

See **Figure 2** for the graphical representation.

4. The Role of Key Underlying Parameters

It is important to recognize that $\hat{e}$ and γ are not independent parameters in this model of the firm. They are both functions of ω and ψ, as specified in (7) and (12) above. Making these substitutions in (15) yields $MAC(e)$ in terms of the fundamental parameters of the model:

(16) $$MAC(e) = \frac{(p\psi - 2\omega e)\zeta}{(1+\zeta)\psi^2}$$

We now want to consider the roles that these underlying parameters play.

(a) The role of *p*

Figure 3 illustrates the impact of an increase in p on $\tilde{\pi}(p,e)$. A higher price raises profit at every value of *e* but it also extends the range of *e* over which the constraint is binding because $\hat{e}$ rises. Critically, a higher price also raises the opportunity cost of cutting production to meet the constraint, and this raises the slope of $\tilde{\pi}(p,e)$ at every value of *e*. The impact on $MAC(e)$ is a parallel shift, as illustrated in **Figure 4.**

Construct plots like those in Figures 3 and 4 and label them accordingly in your output.

(b) The role of ω

Figure 5 illustrates the impact of an increase in ω on $\tilde{\pi}(p,e)$. Profit is lower at every value of *e*, and $\hat{e}$ falls. Output is now has lower net value so the opportunity cost of cutting production to meet the constraint also falls, and this reduces the slope of $\tilde{\pi}(p,e)$ at every value of $e > 0$. However, this effect is smaller at lower values of *e* because the effect of ω on profit is proportional to $(1+\zeta x)^2$, and x is decreasing in ω at any given value of *e*. (In contrast, the effect of *p* on profit does not depend directly on x). At $e = 0$, $x = 1$ at any value of $y > 0$, so a change in ω has no effect on the slope of $\tilde{\pi}(p,e)$ at that point. Thus, the impact of an increase in ω on $MAC(e)$ is an inward pivot around the vertical intercept, as illustrated in **Figure 6**.

Construct plots like those in Figures 5 and 6 and label them accordingly in your output.

(c) The role of ζ

Figure 7 illustrates the impact of an increase in ζ on $\tilde{\pi}(p,e)$. Profit is lower at every value of *e* where the constraint binds, but $\hat{e}$ is unchanged. It is now more costly to adopt a cleaner production process, and so meeting the constraint is also more costly. This raises the slope of $\tilde{\pi}(p,e)$ at every value of $e > 0$. This effect is larger at lower values of *e* because $(1+\zeta x)^2$ is strictly convex in *x*. At $e = \hat{e}$, $x = 0$, so a change in ψ has no effect on the slope of $\tilde{\pi}(p,e)$ at that point. Thus, the impact of an increase in ψ on $MAC(e)$ is an upward pivot around the horizontal intercept, as illustrated in **Figure 8**.

Construct plots like those in Figures 7 and 8 and label them accordingly in your output.

(d) The role of ψ

Figure 9 illustrates the impact of an increase in ψ on $\tilde{\pi}(p,e)$. There are three key points to note from the figure. First, ψ has no effect on unregulated profit (unlike *p* and ω). Second, the increase in ψ raises $\hat{e}$. Third, regulated profit is decreasing in ψ unless $e = 0$ (where $x = 1$ and so emissions are zero for any ψ). Together these three effects mean that the direction of the impact of ψ on the slope of $\tilde{\pi}(p,e)$ depends on *e*. In particular, an increase in ψ causes a counter-clockwise pivot of $MAC(e)$ around an interior point, as illustrated in **Figure 10**. The value of *e* at this pivot point is

(17) $$\bar{e} = \frac{p\psi_1\psi_2}{2\omega(\psi_1 + \psi_2)}$$

where $MAC(e)_1$ and $MAC(e)_2$ cross.

Construct plots like those in Figures 9 and 10 and label them accordingly in your output.

5. Optimal Emissions under Two Different Technologies

Suppose that the damage function is

(18) $$D(e) = \frac{\delta e^2}{2}$$

(a) Optimal Emissions under a "Dirty" Technology

Consider a setting where $\psi = 1$. Derive the optimal level of emissions in this setting.

(b) Optimal Emissions under a "Cleaner" Technology

Now consider a setting where $\psi < 1$. Derive the optimal level of emissions in this setting.

(c) Are Optimal Emissions Always Lower under a Cleaner Technology?

Derive a condition on δ under which optimal emissions are higher under the cleaner technology, and relate your answer to the pivot point identified in (18) above.

APPENDIX

Pollution Control via the Diversion of Inputs

Consider a standard Cobb-Douglas production function with *m* inputs, $z_1,...,,z_m$:

(A1) $$f(z) = \prod_{j=1}^{m} z_j^{a_j}$$

Normalize units such that the emissions-intensity of this production is

(A2) $$\varepsilon_0 \equiv \frac{e}{f(z)} = 1$$

Now suppose the firm can reduce the emissions from its facility by diverting a fraction $\rho_j \in [0,1)$ of input *j* away from actual production and into pollution-control. Output is thereby reduced to

(A3) $$h(z,\rho) = \prod_{j=1}^{m} ((1-\rho_j)z_j)^{a_j}$$

The Cost-Minimization Problem

The cost-minimization problem at any given choice of ρ:

(A4) $$\min_z \sum_{j=1}^{m} w_j z_j \quad \text{subject to} \quad y = \prod_{j=1}^{m} ((1-\rho_j)z_j)^{a_j}$$

where $\{w_j\}$ are the input prices. The first-order condition for input *k* is

(A5) $$w_k = \lambda a_k (1-\rho_k) z_k^{a_k - 1} \prod_{j \neq k} ((1-\rho_j)z_j)^{a_j} \quad \forall k$$

where λ is the Lagrange multiplier. Upon substitution of the constraint into (A5) we obtain

(A6) $$w_k = \frac{\lambda a_k y}{z_k}$$

Rearrange (A6) to make z_k the subject, multiply both sides by $(1-\rho_k)$, and raise both sides to the power a_k to obtain

(A7) $$((1-\rho_k)z_k)^{a_k} = \left(\frac{\lambda a_k (1-\rho_k) y}{w_k} \right)^{a_k}$$

Now take the product of both sides from $k = 1,...,m$ to obtain

(A8) $$y = \prod_{k=1}^{m} \left(\frac{\lambda a_k (1-\rho_k) y}{w_k} \right)^{a_k}$$

We can now solve for λ:

(A9) $$\hat{\lambda} = y^{\frac{1-A}{A}} W^{\frac{1}{A}}$$

where

(A10) $$A = \sum_{k=1}^{m} a_k$$

and

(A11) $$W(\rho) = \prod_{k=1}^{m} \left(\frac{w_k}{a_k (1-\rho_k)} \right)^{a_k}$$

We can now find $\hat{z}_k$ from (A6):

(A12) $$\hat{z}_k = \left(\frac{a_k}{w_k} \right) (yW(\rho))^{\frac{1}{A}} \quad \forall k$$

Now substitute these conditional demands into cost to find the cost function:

(A13) $$c(y, \rho) = \sum_{k=1}^{m} w_k \hat{z}_k = \sum_{k=1}^{m} a_k (yW(\rho))^{\frac{1}{A}} = A(yW(\rho))^{\frac{1}{A}}$$

We will henceforth restrict attention to a special case in which $\rho_j = \rho \quad \forall j$. That is, pollution control requires that all inputs be diverted in the same proportion.

The Intensity Function

Suppose that the diversion of inputs into pollution control reduces the emissions-intensity of production in such a way that

(A14) $$\varepsilon(\rho) \equiv \frac{e}{h(z, \rho)} = 1 - \alpha \left(\frac{\rho}{1-\rho} \right)^{\frac{1}{2}}$$

This relationship between $\varepsilon(\rho)$ and ρ is plotted in **Figure A1**. The figure tells us that the diversion of inputs initially has a significant impact on emissions-intensity as the easiest pollution-control measures are undertaken first. Further diversion has diminishing returns – the rate of decline in $\varepsilon(\rho)$ starts to flatten out – because finding additional cuts to pollution becomes more difficult once the easiest ones have already been made. However, as an increasingly greater fraction of inputs is diverted away from production and into pollution control, emissions-intensity eventually begins to fall rapidly again because production itself is falling towards zero but emissions are falling towards zero faster. Emissions eventually fall to zero at

(A15) $$\bar{\rho} = \frac{1}{1+\alpha^2}$$

Now define a new variable

(A16 $$x(\rho) \equiv 1 - \varepsilon(\rho) = \alpha\left(\frac{\rho}{1-\rho}\right)^{\frac{1}{2}}$$

and express ρ in terms of *x*:

(A17) $$\rho = \frac{x^2}{\alpha^2 + x^2}$$

Substitution into (A11) with $\rho_j = \rho \ \ \forall j$ yields

(A18) $$W(x) = \left(1 + \frac{x^2}{\alpha^2}\right)^A \prod_{k=1}^{m}\left(\frac{w_k}{a_k}\right)^{a_k}$$

and so the cost function from (A13) becomes

(A19) $$c(y,x) = A(yW(x))^{\frac{1}{A}}$$

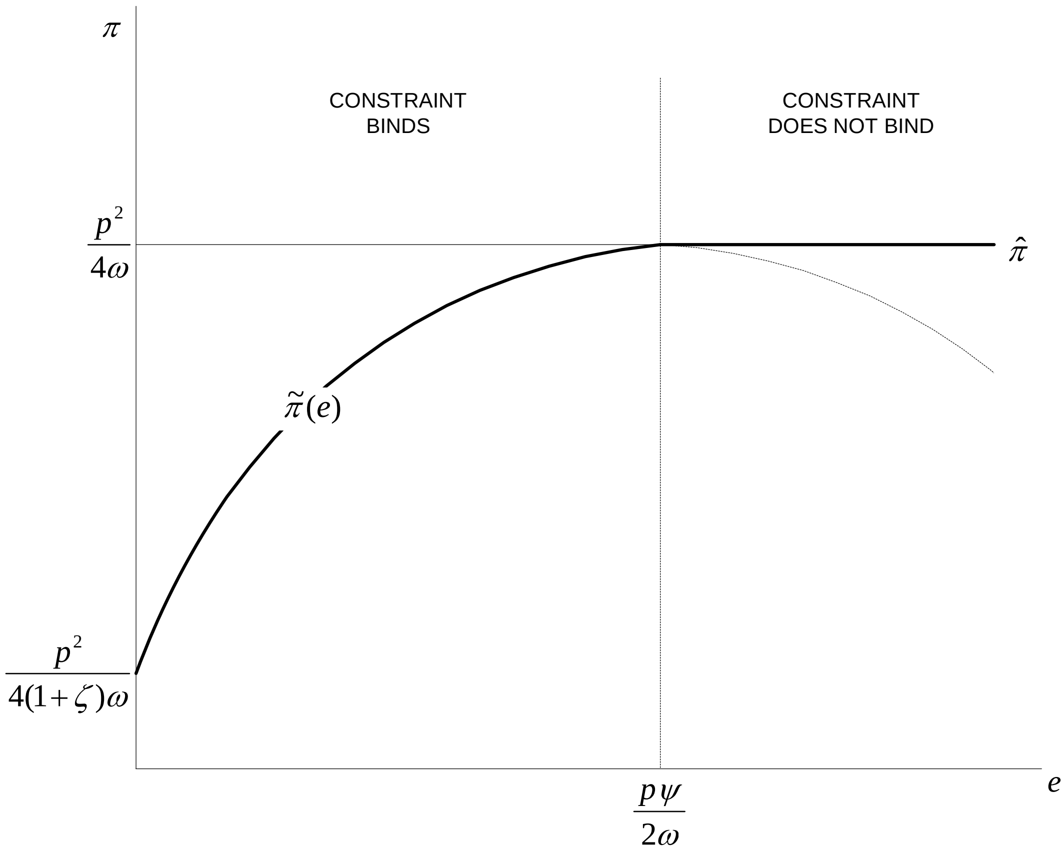
Special Case: Quadratic Costs

We now impose additional restrictions that yield the simple quadratic cost function in the text. In particular, we set $A = \frac{1}{2}$ and define $\zeta \equiv \alpha^{-2}$. This yields

(A20) $$c(y,x) = \omega(1+\zeta x^2)y^2$$

where

(A21) $$\omega = \frac{1}{2}\left(\prod_{k=1}^{m}\left(\frac{w_k}{a_k}\right)^{a_k}\right)^2$$



π
CONSTRAINT BINDS
CONSTRAINT DOES NOT BIND
$\frac{p^2}{4\omega}$
$\hat{\pi}$
$\tilde{\pi}(e)$
$\frac{p^2}{4(1+\zeta)\omega}$
e
$\frac{p\psi}{2\omega}$

Figure 1

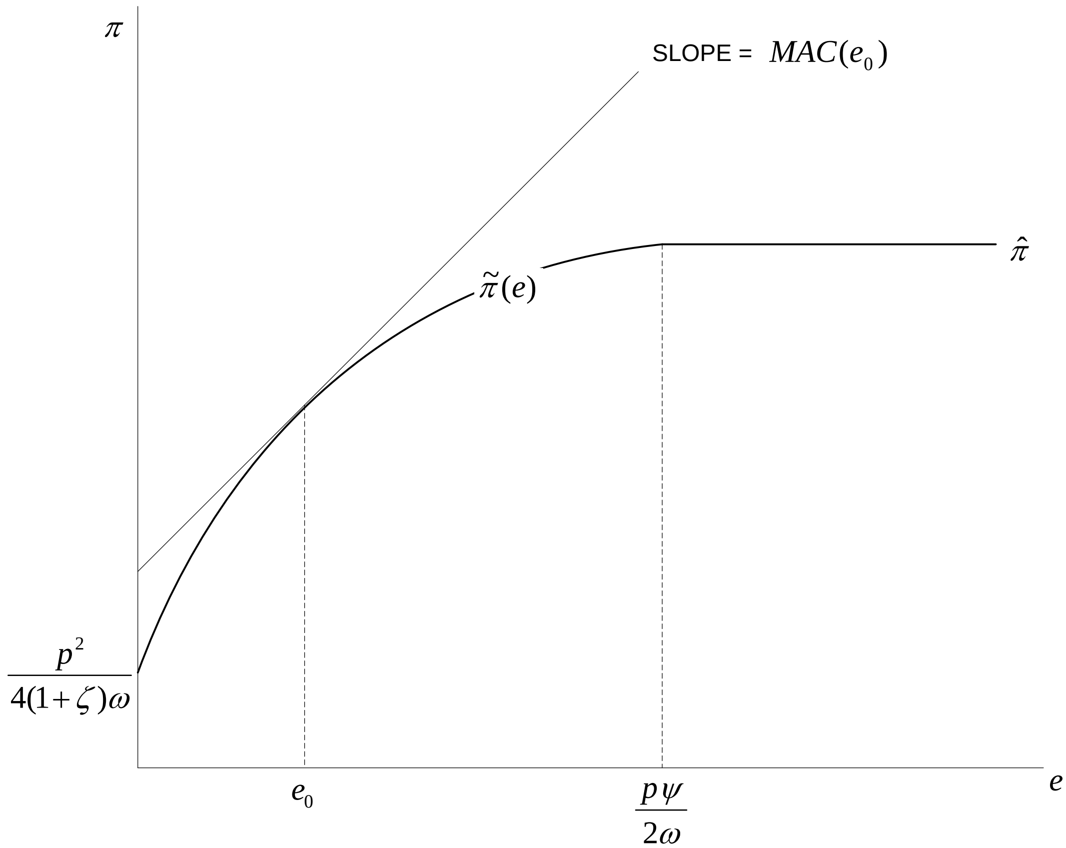

π
SLOPE = $MAC(e_0)$
$\hat{\pi}$
$\tilde{\pi}(e)$
$\frac{p^2}{4(1+\zeta)\omega}$
e_0
$\frac{p\psi}{2\omega}$
e

Figure 2

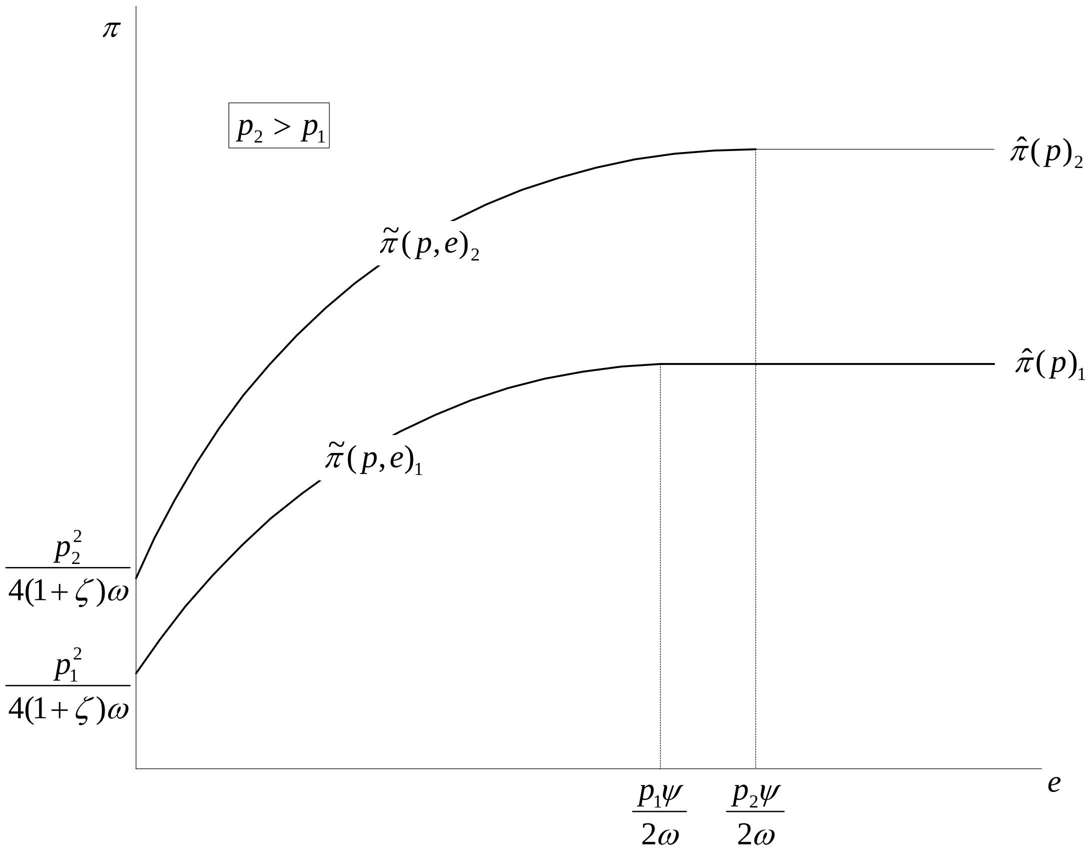

π
$p_2 > p_1$
$\hat{\pi}(p)_2$
$\tilde{\pi}(p,e)_2$
$\hat{\pi}(p)_1$
$\tilde{\pi}(p,e)_1$
$\frac{p_2^2}{4(1+\zeta)\omega}$
$\frac{p_1^2}{4(1+\zeta)\omega}$
e
$\frac{p_1\psi}{2\omega}$
$\frac{p_2\psi}{2\omega}$

Figure 3

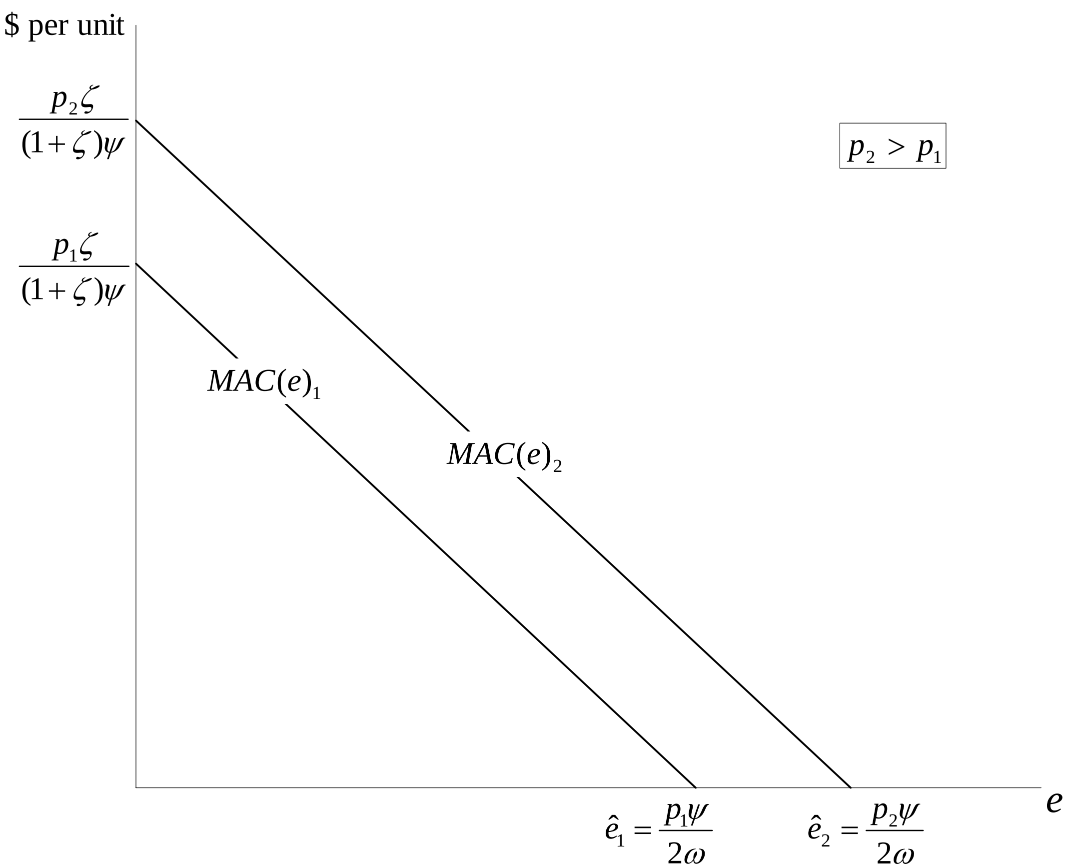
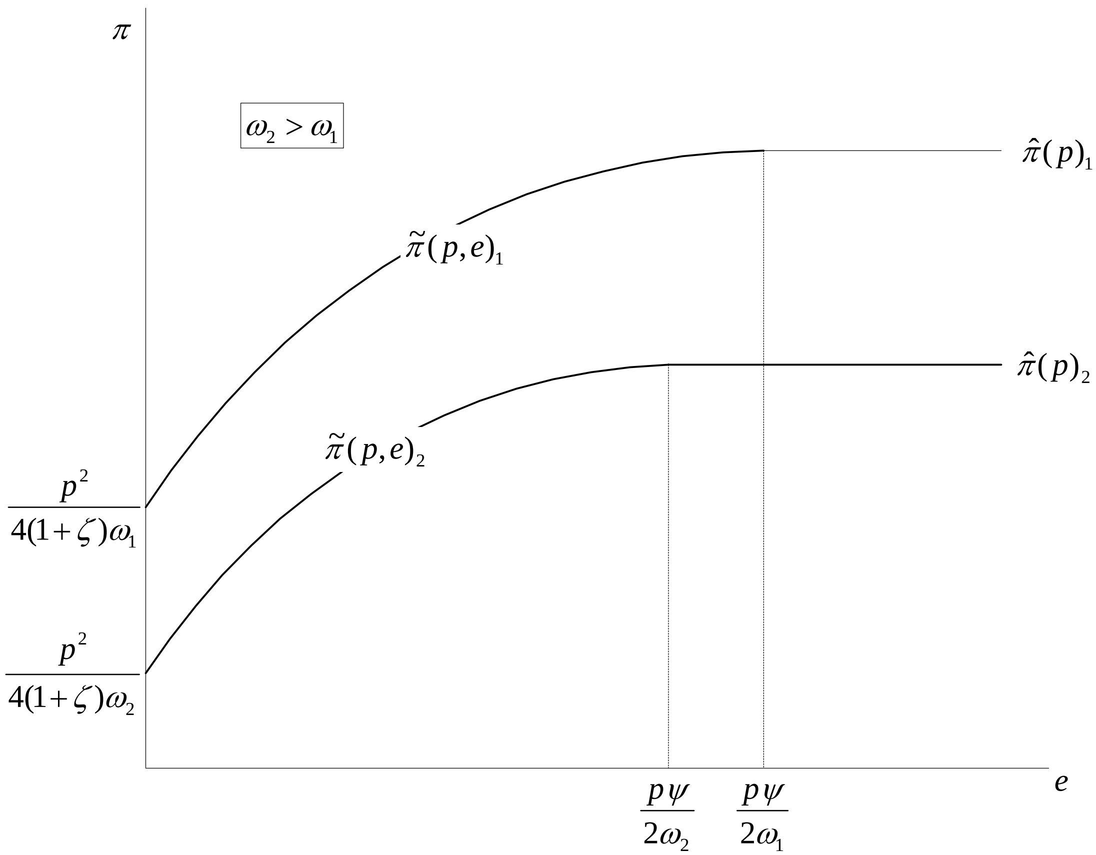

$ per unit
$\frac{p_2\zeta}{(1+\zeta)\psi}$
$\frac{p_1\zeta}{(1+\zeta)\psi}$
$p_2 > p_1$
$MAC(e)_1$
$MAC(e)_2$
e
$\hat{e}_1 = \frac{p_1\psi}{2\omega}$
$\hat{e}_2 = \frac{p_2\psi}{2\omega}$

Figure 4



π
$\omega_2 > \omega_1$
$\hat{\pi}(p)_1$
$\tilde{\pi}(p,e)_1$
$\hat{\pi}(p)_2$
$\tilde{\pi}(p,e)_2$
$\frac{p^2}{4(1+\zeta)\omega_1}$
$\frac{p^2}{4(1+\zeta)\omega_2}$
e
$\frac{p\psi}{2\omega_2}$
$\frac{p\psi}{2\omega_1}$

Figure 5

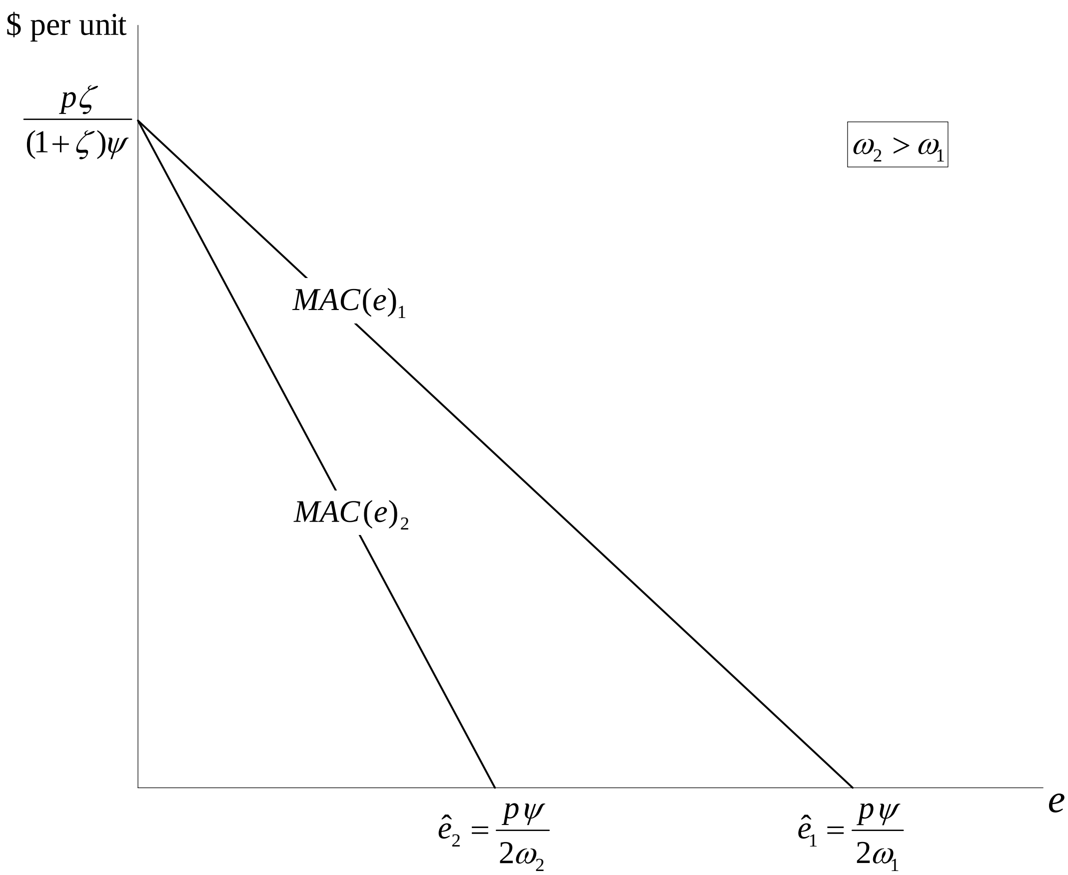
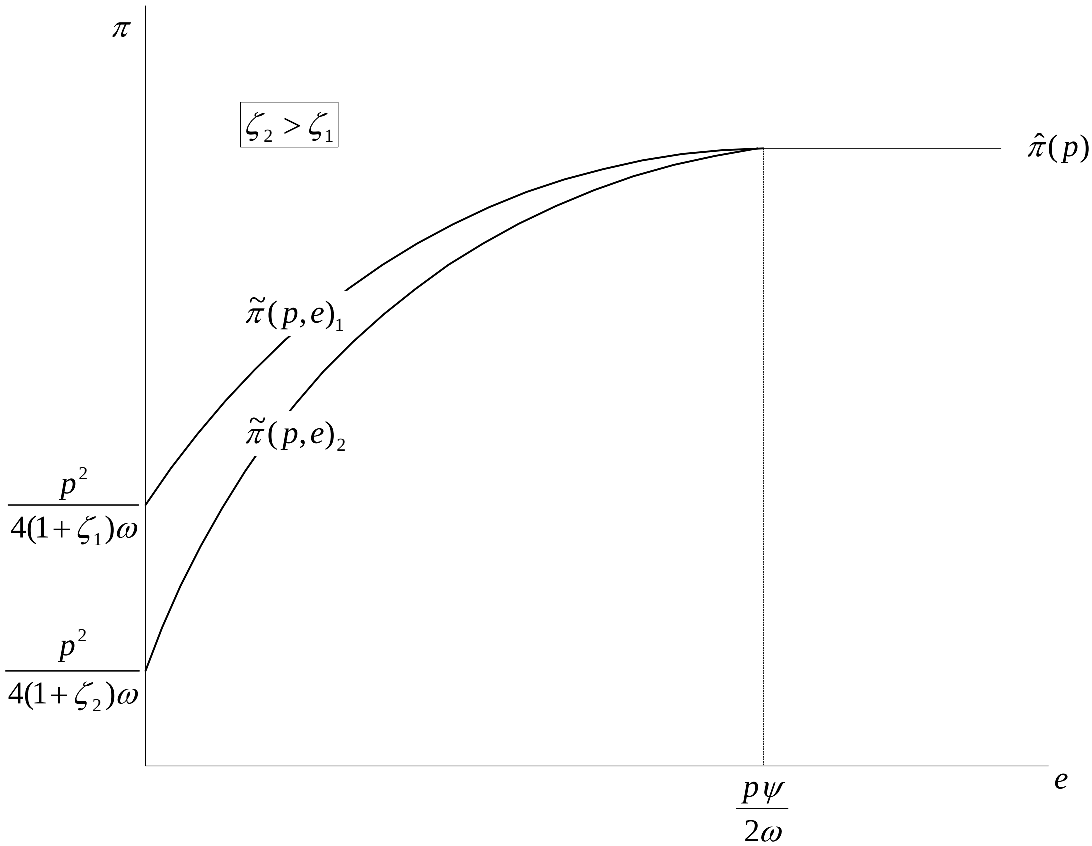

$ per unit
$\frac{p\zeta}{(1+\zeta)\psi}$
$\omega_2 > \omega_1$
$MAC(e)_1$
$MAC(e)_2$
e
$\hat{e}_2 = \frac{p\psi}{2\omega_2}$
$\hat{e}_1 = \frac{p\psi}{2\omega_1}$

Figure 6



π
$\zeta_2 > \zeta_1$
$\hat{\pi}(p)$
$\tilde{\pi}(p,e)_1$
$\tilde{\pi}(p,e)_2$
$\frac{p^2}{4(1+\zeta_1)\omega}$
$\frac{p^2}{4(1+\zeta_2)\omega}$
e
$\frac{p\psi}{2\omega}$

Figure 7

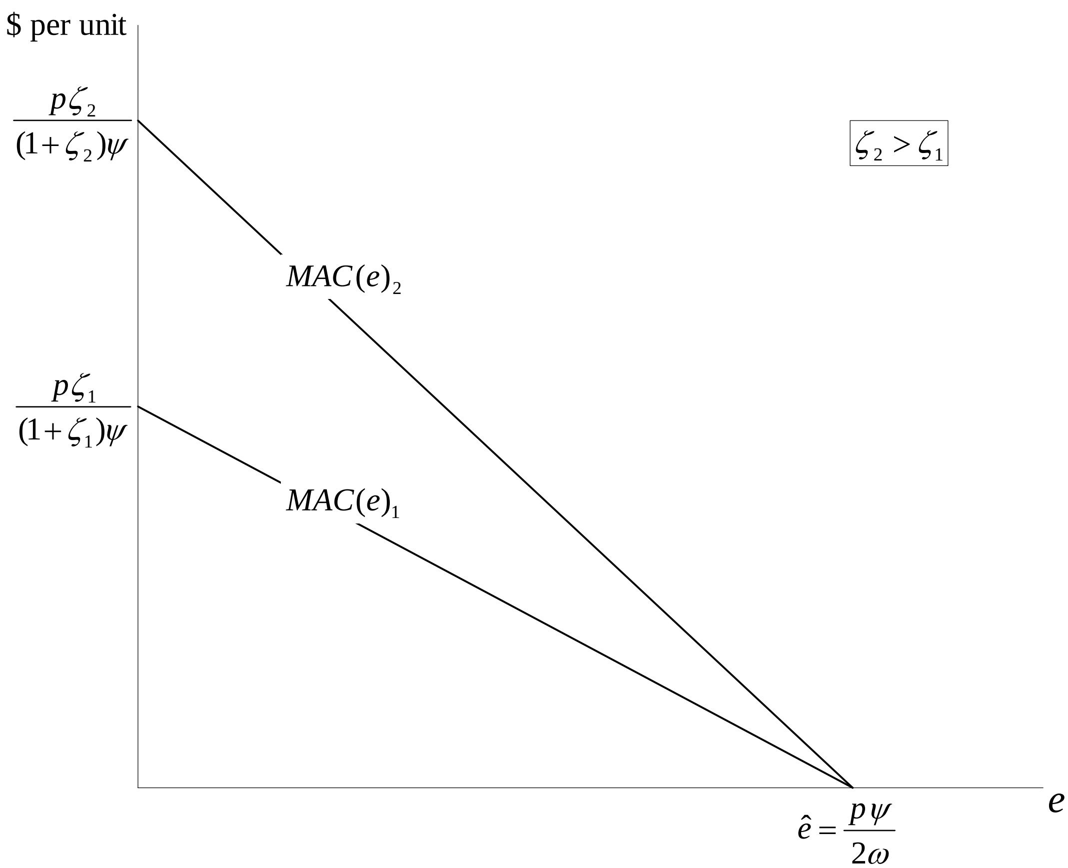
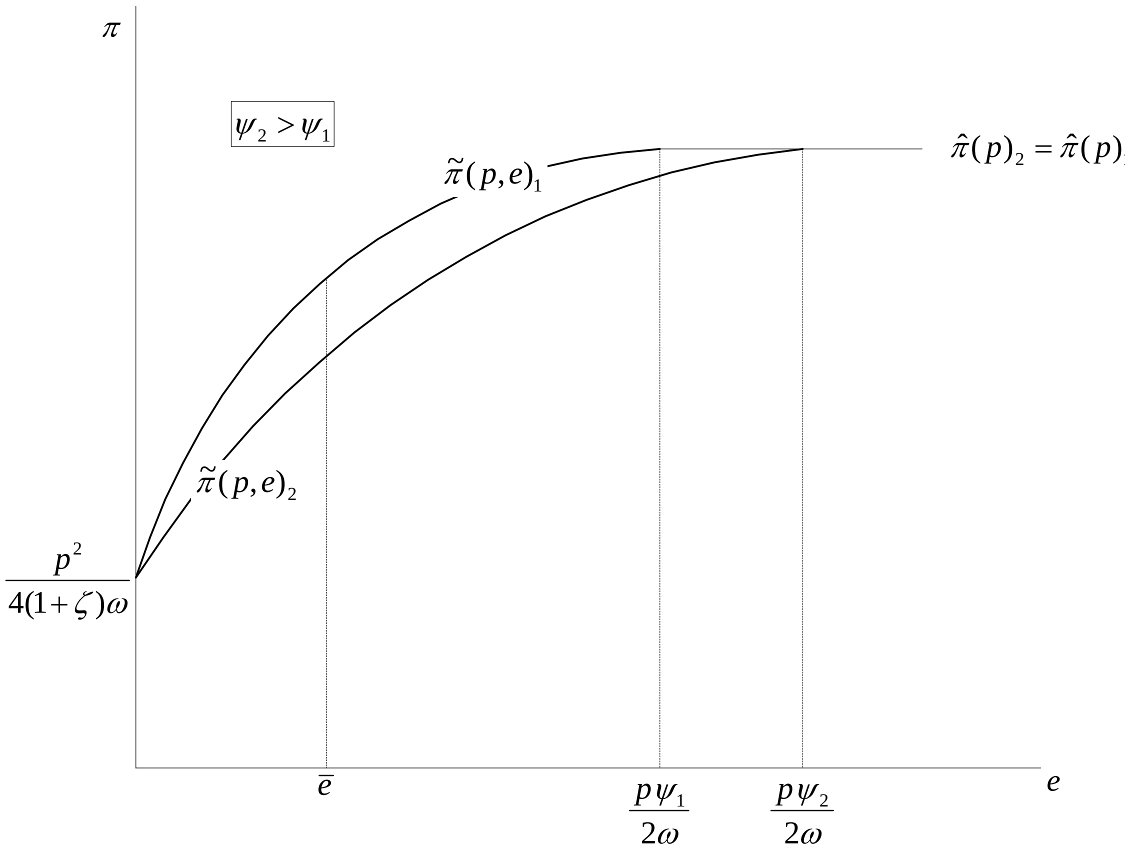

$ per unit
$\frac{p\zeta_2}{(1+\zeta_2)\psi}$
$\zeta_2 > \zeta_1$
$MAC(e)_2$
$\frac{p\zeta_1}{(1+\zeta_1)\psi}$
$MAC(e)_1$
e
$\hat{e} = \frac{p\psi}{2\omega}$

Figure 8



π
$\psi_2 > \psi_1$
$\tilde{\pi}(p,e)_1$
$\hat{\pi}(p)_2 = \hat{\pi}(p)_1$
$\tilde{\pi}(p,e)_2$
$\frac{p^2}{4(1+\zeta)\omega}$
$\bar{e}$
$\frac{p\psi_1}{2\omega}$
$\frac{p\psi_2}{2\omega}$
e

Figure 9

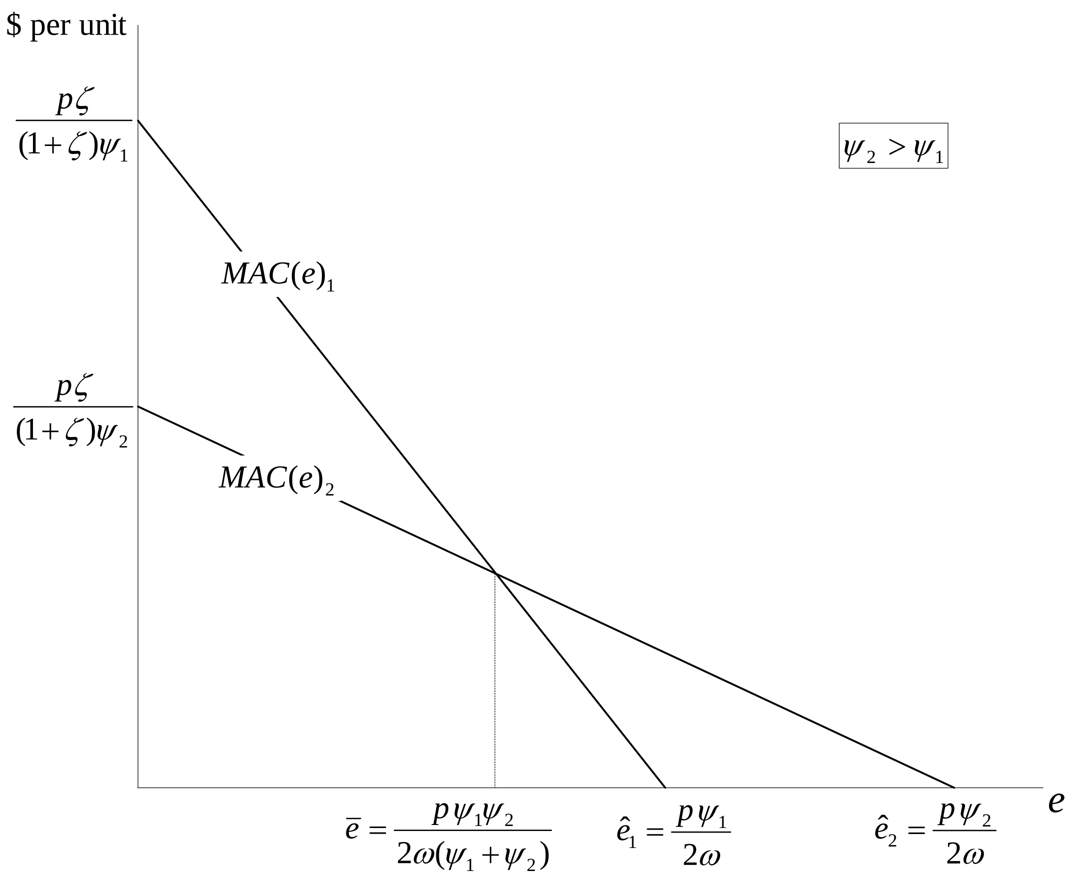

$ per unit
$\frac{p\zeta}{(1+\zeta)\psi_1}$
$\psi_2 > \psi_1$
$MAC(e)_1$
$\frac{p\zeta}{(1+\zeta)\psi_2}$
$MAC(e)_2$
e
$\bar{e} = \frac{p\psi_1\psi_2}{2\omega(\psi_1+\psi_2)}$
$\hat{e}_1 = \frac{p\psi_1}{2\omega}$
$\hat{e}_2 = \frac{p\psi_2}{2\omega}$

Figure 10

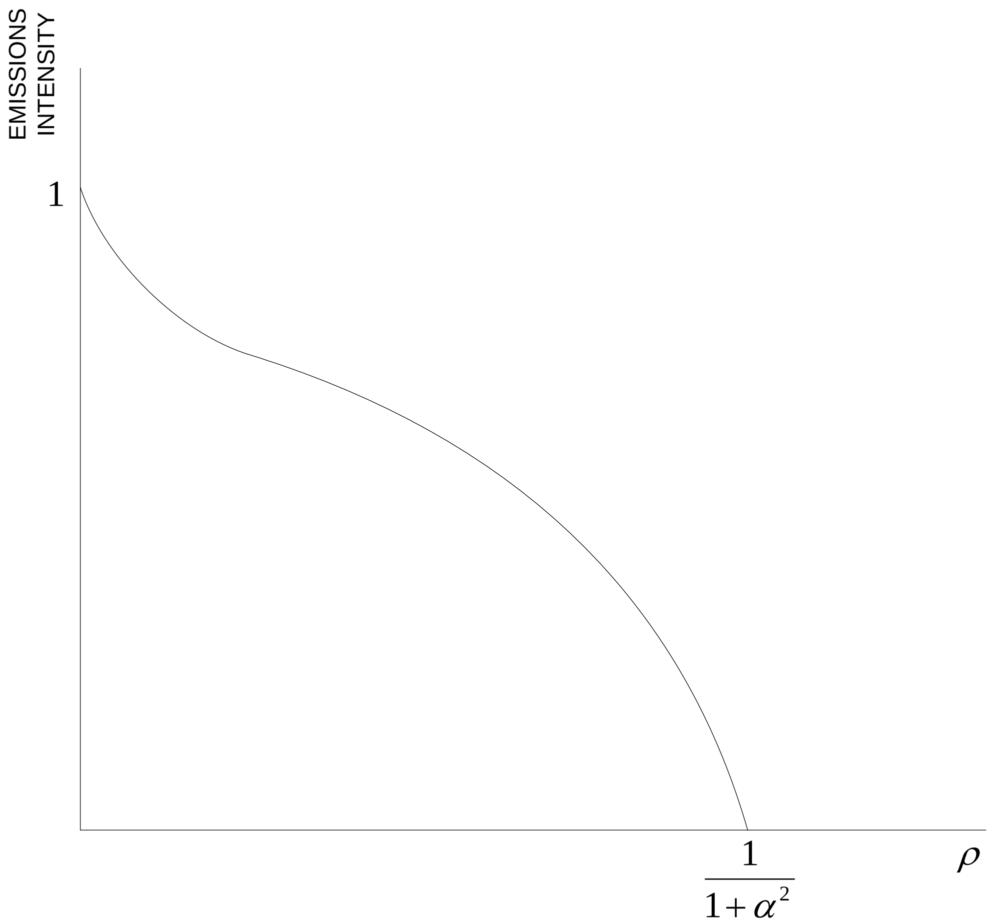


Figure A1