3.6 Homothetic Preferences

Technical Note. Suppose a function $g(x)$ is homogeneous of degree α, such that

$$g(\theta x) = \theta^{\alpha} g(x) \quad \forall \theta$$

If $f(x) = F(g(x))$, where F is a monotonic function, then $f(x)$ is homothetic. That is, a function is homothetic if it can be expressed as a monotonic transform of a homogeneous function.

Preferences are said to be homothetic if they can be represented by a homothetic utility function. What are the implications of this property?

We know that if some function $u(x)$ is a representation of preferences then any monotonic transform of that function is also a representation of those same preferences (because utility has no cardinal units). Thus, if $u(x)$ is a homothetic function then we can take a monotonic transform of $u(x)$ to obtain a homogeneous function that represents the same preferences. That is, we can recover the homogeneous function on which the homothetic function is based.

Moreover, we can always choose a transform that ensures that the transformed utility function is homogeneous of degree 1. Thus, if preferences can be represented by a homothetic function, then they can also be represented by a function that is homogeneous of degree 1.

For example, the log-linear utility function

$$\log(u(x)) = a \log x_1 + b \log x_2$$

is a homothetic function because we can construct it as a monotonic transform of a homogeneous function. In particular, the Cobb-Douglas function

$$u(x) = x_1^a x_2^b$$

is a homogenous function of degree $(a + b)$. The log transform of that function is

$$\log(u(x)) = a \log x_1 + b \log x_2$$

and this transform is monotonic. Thus, the log-linear utility function is homothetic by definition.

Now suppose we start with this homothetic log-linear function and transform it to recover the underlying homogenous function:

$$u(x) = x_1^a x_2^b$$

Now transform it again by raising it to the power $1/(a+b)$, to yield

$$u(x) = x_1^{\frac{a}{a+b}} x_2^{\frac{b}{a+b}}$$

Now define $\gamma = a/(a+b)$. We can now write our transformed function as

$$u(x) = x_1^{\gamma} x_2^{1-\gamma}$$

and this is homogenous of degree 1. Thus, we are able to transform a homothetic utility function into a function that is homogeneous of degree 1 through a series of monotonic transforms that do not change the underlying preferences.

What are the implications of this transformability?

If $u(x)$ is homogeneous of degree 1 then

$$u(\theta x) = \theta u(x)$$

Consider the MRS for such a function, evaluated at some bundle $\tilde{x}$:

$$MRS(\tilde{x}) = -\frac{\partial u(\tilde{x})}{\partial x_1} \Bigg/ \frac{\partial u(\tilde{x})}{\partial x_2}$$

and at some other bundle proportional to $\tilde{x}$:

$$MRS(\theta\tilde{x}) = -\frac{\partial u(\theta\tilde{x})}{\partial x_1} \Bigg/ \frac{\partial u(\theta\tilde{x})}{\partial x_2} = -\frac{\theta \partial u(\tilde{x})}{\partial x_1} \Bigg/ \frac{\theta \partial u(\tilde{x})}{\partial x_2} = MRS(\tilde{x})$$

That is, the MRS is the same at both bundles. This means that MRS is constant along a ray; see Figure 3.9. This is the defining feature of homothetic preferences.

The key implication of this property is that Marshallian demands are linear in income. To see why, note that if the utility function is homogenous of degree 1 then the expenditure function is linear in *u*. That is, it can be written in the form

$$e(p,u) = a(p)u$$

Why? To double *u* we must double *x* (with no change in consumption proportions) since $u(\theta x) = \theta u(x)$, and to double *x* (at given prices) we must double expenditure.

Now consider $v(p,m)$. Set $e(p,u) = m$ and $v(p,m) = u$ to obtain

$$m = a(p)v(p,m)$$

Thus, we can write

$$v(p,m) = \frac{m}{a(p)} \equiv b(p)m$$

where $b(p) = 1/a(p)$ is introduced to simplify the differentiation that follows.

By Roy's Identity:

$$x_i(p,m) = -\frac{\partial v(p,m)}{\partial p_i} \Big/ \frac{\partial v(p,m)}{\partial m} = \frac{m}{b(p)} \frac{\partial b}{\partial p_i}$$

Thus, Marshallian demands associated with homothetic preferences are linear in income. (Equivalently, the Engel curves – which plot x_i against *m* – are linear).

Why does this matter? If all agents have identical homothetic preferences then we can aggregate across those individuals to construct an aggregate demand that is a function of aggregate income; the *distribution* of that aggregate income across those agents has no bearing on the aggregate demand. This simplifies the analysis of many trade and macroeconomic models (albeit at the cost of ignoring a key issue in real economies).