ADDENDUM TO PROBLEM SET 2

Question 4B

An agent has the following indirect utility function:

$$v(m) = \log(m)$$

where the functional dependence of $v(.)$ on p has been suppressed for notational simplicity.

The agent's current income is \$200 but there is a 10% chance that she will have to switch jobs, in which case her income will fall to \$100.

(a) Find the certainty-equivalent income level and the risk premium associated with this prospect.

(b) Suppose she can purchase insurance against this income fall at price $\frac{1}{7}$ per dollar of coverage. How much insurance will she purchase?

Question 4C

An agent has the following indirect utility function:

$$v(m) = m^{\frac{1}{2}}$$

where the functional dependence of $v(.)$ on p has been suppressed for notational simplicity.

The agent's current salary income is \$6300 but there is a 25% chance that this salary income will fall to \$300. She also receives a non-salary income F that is not subject to uncertainty.

(a) For this part only, suppose $F = 100$. Find the certainty-equivalent income level and the risk premium associated with this prospect.

(b) Now suppose she can purchase insurance against this income fall at an actuarially fair price. How much insurance will she purchase? Explain the relationship between this value and *F*.

(c) Now suppose that she pay price $\frac{1}{2}$ per dollar of insurance purchased. At what value of *F* does she buy no insurance? Explain why the value of *F* matters here.

Question 4D

An agent has the following utility function

$$u(x_1, x_2) = a\log(x_1) + b\log(x_2)$$

He has income *m*, and faces prices p_1 and p_2.

(a) Find his indirect utility function. (Do not take a transformation of the utility function).

(b) Suppose he faces an uncertain price for good 1. In particular, $p_1 = p_{1H}$ with probability π, and $p_1 = p_{1L}$ with probability $1-\pi$. He must commit to his purchase choices before the uncertainty is resolved. What certain price would make him just indifferent between this certain price and the uncertain prospect?

(c) The price from part (b) is the certainty-equivalent price. How does this compare with the expected price?

Question 5B

An agent has the following intertemporal utility function:

$$u(c_1, c_2) = c_1\left(\frac{c_2}{1+\rho}\right)$$

where c_1 is current consumption and c_2 is future consumption. The agent has income profile $\{y_1, y_2\}$. Do not transform the utility function.

Suppose the agent can neither borrow nor lend on the market but he can store some of his income from period 1 to finance consumption in period 2.

(a) Find a condition on y_2 relative to y_1 under which he will store some of his income (that is, save).

(b) Assume that he does save. Find the indirect utility function. Let $v(y_1, y_2)^A$ denote this indirect utility (where "A" indicates autarky; that is, "no trade").

Now suppose the agent can borrow and lend at interest rate *r*, but where *r* is uncertain. In particular, $r = r_H > 0$ with probability π, and $r = r_L < 0$ with probability $1-\pi$.

(c) Find a condition on π under which he will save.

(d) Assume that when faced with the uncertain prospect, he would choose to save. Find the indirect utility function. Let $v(y_1, y_2)^T$ denote this indirect utility (where "T" indicates "trade").

(e) Now consider the choice between autarky and trade. Find a condition on π under which our agent will choose trade over autarky.

ANSWER GUIDE

Question 4B

(a) The certainty-equivalent wealth is the solution to an indifference equation:

$$v(\hat{m}) = \mathrm{E}[v(m)]$$

In this case,

$$v(\hat{m}) = \log(\hat{m})$$

$$\mathbf{E[}v(m)] = \pi \log(m_1) + (1-\pi)\log(m_2)$$

Solving for $\hat{m}$ yields

$$\hat{m} = m_1^{\pi} m_2^{1-\pi}$$

For the values in the Question: $m_1 = 100$, $m_2 = 200$ and $\pi = 0.1$. Thus, $\hat{m} = 186.61$

Expected wealth is

$$\mathbf{E[}m] = \pi m_1 + (1-\pi)m_2$$

For the values in the Question: $\mathbf{E[}m] = 190$

The risk premium is

$$R \equiv \mathrm{E}[m] - \hat{m}$$

For the values in the Question: $R = 3.39$

(b) Expected utility with insurance is

$$\mathbf{E[}v(m;q)] = \pi \log(m_1 - rq + q) + (1-\pi)\log(m_2 - rq)$$

Differentiate with respect to q and solve to yield

$$q(r) = \frac{\pi m_2}{r} - \frac{(1-\pi)m_1}{1-r}$$

For the values in the Question: $q = 35$

Question 4C

(a) The certainty-equivalent wealth is the solution to an indifference equation:

$$v(\hat{m}) = \mathrm{E}[v(m)]$$

In this case,

$$v(\hat{m}) = \hat{m}^{\frac{1}{2}}$$

$$\mathbf{E[}v(m)] = \pi(y - L + F)^{\frac{1}{2}} + (1-\pi)(y + F)^{\frac{1}{2}}$$

where y is her salary without loss.

The solution for $\hat{m}$ is messy but it is simple to solve with the numerical values in place:

$$\hat{m} = 4225$$

Expected wealth is

$$\mathbf{E[}m] = \pi(y - L + F) + (1-\pi)(y + F)$$

For the values in the Question: $\mathbf{E[}m] = 4900$

The risk premium is

$$R \equiv \mathrm{E}[m] - \hat{m}$$

For the values in the Question: $R = 675$

(b) Expected utility with insurance is

$$\mathbf{E[}v(m;q)] = \pi(y - L + F - rq + q)^{\frac{1}{2}} + (1-\pi)(y + F - rq)^{\frac{1}{2}}$$

In principle, differentiate with respect to q and solve to yield $q(r, F)$. The solution is messy but it is simple to solve with the numerical values in place. For $r = \pi = \frac{1}{4}$, $q = L$. That is, full insurance.

(c) For $r = \frac{1}{2}$, $q = 450 - F$. Thus, if $F \geq 450$ then she buys no insurance (because that fixed, certain back-up effectively provides some insurance against loss).

Question 4D

(a) This is a standard log-linear utility function, and we know that the associated Marshallian demands

$$x_1(p,m) = \frac{am}{(a+b)p_1} \quad \text{and} \quad x_2(p,m) = \frac{bm}{(a+b)p_2}$$

The indirect utility function is

$$\begin{aligned} v(p,m) &= a\log(x_1(p,m)) + b\log(x_2(p,m)) \\ &= a\log\left(\frac{am}{(a+b)p_1}\right) + b\log\left(\frac{bm}{(a+b)p_2}\right) \end{aligned}$$

(b) The agent faces uncertainty about p_1 so we want to express $v(p,m)$ in a way that allows us to isolate its dependence on p_1 in the simplest possible form. We do this by opening up the first log term to pull out $\log(p_1)$, yielding

$$v(p,m) = a\log\left(\frac{am}{(a+b)}\right) + b\log\left(\frac{bm}{(a+b)p_2}\right) - a\log(p_1)$$

and this can expressed

$$v(p_1) = k - a\log(p_1)$$

where k is a term that does not depend on p_1, and hence is not subject to uncertainty.

The certainty-equivalent value of p_1 solves the following indifference equation:

$$\mathbf{E}[v(p_1)] = v(\hat{p}_1)$$

In this case,

$$v(\hat{p}_1) = k - a\log(\hat{p}_1)$$

and

$$\begin{aligned}\mathbf{E}[v(p_1)] &= \pi(k - a\log(p_{1H})) + (1-\pi)(k - a\log(p_{1L})) \\ &= k - a(\pi\log(p_{1H}) + (1-\pi)\log(p_{1L})) \\ &= k - a(\log(p_{1H}^{\pi}) + \log(p_{1L}^{(1-\pi)})) \\ &= k - a\log(p_{1H}^{\pi}\, p_{1L}^{(1-\pi)})\end{aligned}$$

Setting $\mathbf{E}[v(p_1)] = v(\hat{p}_1)$ and solving for $\hat{p}_1$ yields

$$\hat{p}_1 = p_{1H}^{\pi}\, p_{1L}^{(1-\pi)}$$

(c) The expected price is

$$\mathbf{E}[p_1] = \pi p_{1H} + (1-\pi)\,p_{1L}$$

Figure P2-4 plots $\mathbf{E}[p_1]$ and $\hat{p}_1$ against π. Note that the certainty-equivalent price is <u>lower</u> than the expected price. That is, this agent would need to receive a <u>discount</u> (not a premium) to accept the certain price over the uncertain price. Why? The indirect utility function is convex in *p* (though nonetheless concave in *m*).

Question 5B

(a) Write the problem as one where the agent chooses how much to save in period 1:

$$\max_{s} \quad (y_1 - s)\left(\frac{y_2 + s}{1+\rho}\right)$$

This solves for

$$s^A = \frac{y_1 - y_2}{2}$$

and this is positive if and only if

$$y_1 > y_2$$

(b) Assuming $s^A > 0$, the indirect utility function is

$$v(y_1, y_2)^A = (y_1 - s^A)\left(\frac{y_2 + s^A}{1+\rho}\right)$$

$$= \frac{(y_1 + y_2)^2}{4(1+\rho)}$$

(c) This problem can become very complicated if we do not think carefully before doing the math. In particular, we could write the problem as

$$\max_{c_1 c_2} \quad \mathbf{E}[u(c_1, c_2)]$$

$$\text{subject to} \quad c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

but this becomes complicated (though it can be solved).

Instead, focus on the case of interest; that is, where the agent saves in period 1. If r were certain, we would then write the saving-choice problem from part (a) as

$$\max_{s} \quad (y_1 - s)\left(\frac{y_2 + s(1+r)}{1+\rho}\right)$$

However, r is not certain and so the choice problem is

$$\max_{s} \quad \mathbf{E}\left[(y_1 - s)\left(\frac{y_2 + s(1+r)}{1+\rho}\right)\right]$$

but this is linear in r, so we can simply replace r with its expected value and solve

$$\max_{s} \quad (y_1 - s)\left(\frac{y_2 + s(1+\mathbf{E}[r])}{1+\rho}\right)$$

This solves for

$$s^T = \frac{1}{2}\left(y_1 - \frac{y_2}{1+\mathbf{E}[r]}\right)$$

and it is positive if and only if

$$\mathbf{E}[r] > \frac{y_2}{y_1} - 1$$

Now note that

$$\mathbf{E}[r] = \pi r_H + (1-\pi) r_L$$

Thus, $s^T > 0$ if and only if

$$\pi > \frac{y_2 - y_1(1+r_L)}{y_1(r_H - r_L)}$$

(d) Assuming $s^T > 0$, the indirect utility function is

$$v(y_1, y_2)^T = (y_1 - s^T)\left(\frac{y_2 + s^T(1+r)}{1+\rho}\right)$$

Note that is a function of *r* (which is a random variable) and of $\mathbf{E}[r]$, via s^T.

(e) The agent will choose trade over autarky if and only if

$$\mathbf{E}[v(y_1, y_2)^T] > v(y_1, y_2)^A$$

Since $v(y_1, y_2)^T$ is linear in *r*, we can find its expectation simply by replacing *r* with its expected value:

$$\mathbf{E}[v(y_1, y_2)^T] = (y_1 - s^T)\left(\frac{y_2 + s^T(1+\mathbf{E}[r])}{1+\rho}\right)$$

Making the substitution for s^T, and simplifying via lots of algebra, yields

$$\mathbf{E}[v(y_1, y_2)^T] = \frac{(y_1(1+\mathbf{E}[r]) + y_2)^2}{4(1+\rho)(1+\mathbf{E}[r])}$$

Setting $\mathbf{E}[v(y_1, y_2)^T] = v(y_1, y_2)^A$ and solving for $\mathbf{E}[r]$ yields two roots:

$$\mathbf{E}[r] = 0$$

and

$$\mathbf{E}[r] = \left(\frac{y_2}{y_1}\right)^2 - 1$$

The second of these roots is negative under conditions where the agent saves; recall part (a). Thus, the relevant threshold is $\mathbf{E}[r]=0$. That is, the agent chooses trade over autarky if and only if $\mathbf{E}[r]>0$.

Could we have discovered this without all the work? Yes! The autarky position is effectively one where the agent trades with a risk-free interest rate equal to zero. Since the indirect utility function in part (c) is linear in *r*, the choice between trade and autarky simply becomes one of whether trade yields an expected rate higher than the risk-free rate, which is zero.

So what is the condition on π that yields $\mathbf{E}[r]=0$? Recall that

$$\mathbf{E}[r]=\pi r_H+(1-\pi)r_L$$

Set this to zero and solve for π:

$$\overline{\pi}=\frac{-r_L}{r_H-r_L}$$

and recall that $r_L<0$.

Key message here: think carefully about a problem before attacking it with math.

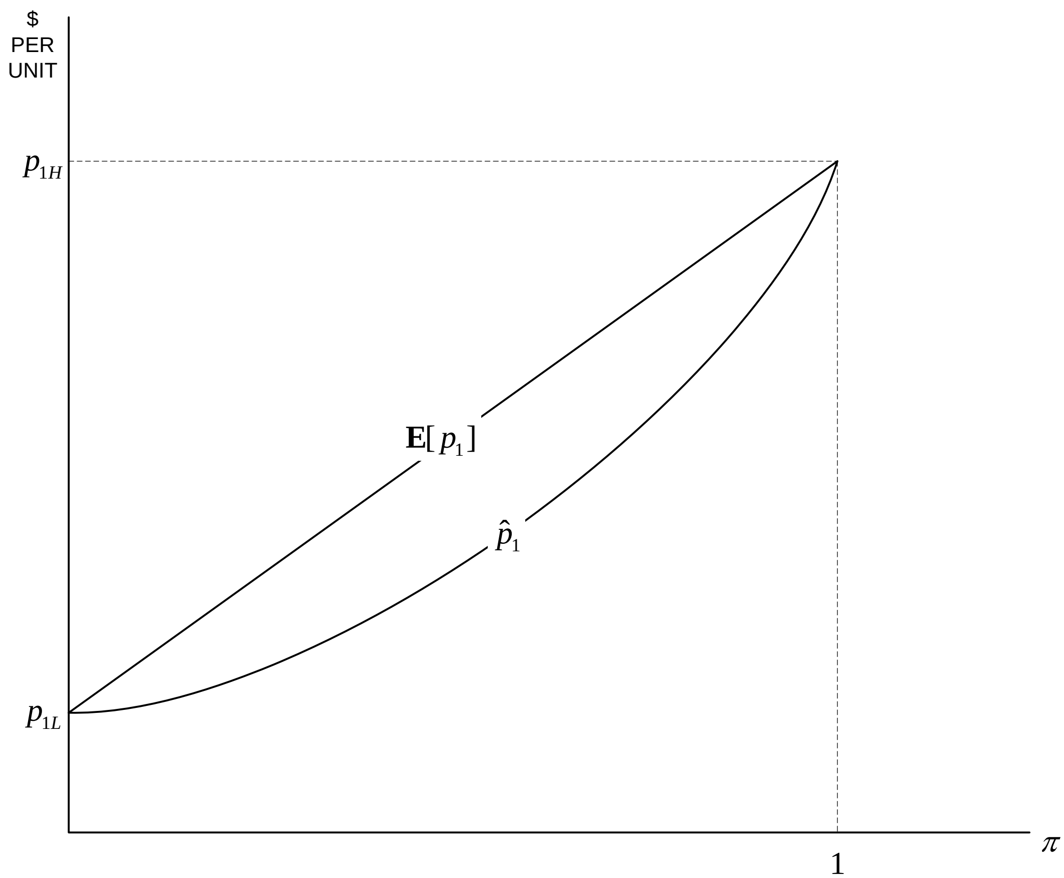

FIGURE P2.4