Introduction to Quantum Computing

Assignment 3 - Due Feb. 12

Unitary operations and quantum circuits

1. **One-qubit rotations.**– The one-qubit "active" rotation operation is defined by

$$R_{\hat{n}}(\theta) = \mathrm{e}^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}} \tag{1}$$

where $\vec{\sigma} = (X, Y, Z)$ denotes the vector of Pauli operators. By "active rotation" we mean that a state (when represented in the Bloch sphere) gets rotated by angle θ about the axis $\hat{n}$ with the direction of the rotation set by the right hand rule.

   (a) Prove the following identity which makes the evaluation of the matrices corresponding to $R_{\hat{n}}(\theta)$ much easier:

$$R_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)\hat{n}\cdot\vec{\sigma}. \tag{2}$$

   (b) Prove the relation:

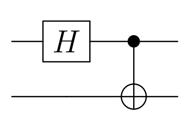
$$R^{\dagger}_{\hat{n}}(\theta) = R_{\hat{n}}(-\theta) = R^{-1}_{\hat{n}}(\theta). \tag{3}$$

2. **Rotation in the Bloch sphere.**– Show that $R_x(\theta)$ applied to $|0\rangle$ produces a state corresponding to the following Bloch vector:

$$\langle\hat{v}\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4}$$

Sketch this operation in the Bloch sphere.

3. **Matrix representation for a circuit.**– What is the 4×4 unitary matrix representing this circuit?



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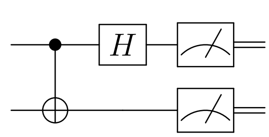
4. **Measurement in the Bell basis.**– Show that this circuit executes measurements in the Bell basis $|\beta_{xy}\rangle$:



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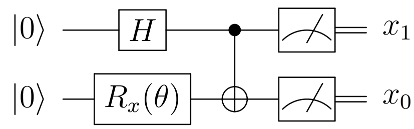
5. **Measurement outcomes of a quantum circuit.**– What are the probabilities $p(x_1, x_2)$ for the following circuit:



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