Lecture 2: Kinetic and Potential Energies, Formal Solution of the 1-D Problem

Newton's Second Law (N_2) for a force that depends on x is

$$F(x) = m\ddot{x}$$

One method to solve this equation formally is to write it as such:

$$\ddot{x} = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

And hence (N_2) becomes

$$F(x) = mv\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}mv^2\right) = \frac{dT}{dx}$$

Where we define $T = \frac{1}{2}mv^2$ as the kinetic energy of a system. We can now express N_2 in integral form:

$$\int_{x_0}^{x} F(x)dx = \int_{x_0}^{x} \frac{dT}{dx}dx = T - T_0$$

The first integral by definition is W (work). This is the work-kinetic energy theorem. We now define the potential energy $V(x)$ as

$$-\frac{dV(x)}{dx} = F(x)$$

(Note that V is defined apart from an arbitrary constant). The work integral becomes

$$\int_{x_0}^{x} F(x)dx = -\int_{x_0}^{x} \frac{dV}{dx}dx = -(V - V_0) = T - T_0$$

$$\implies T + V = T_0 + V_0 = \text{constant} \equiv E$$

We have just proven the conservation of energy theorem. Note that an important assumption we made is that the force F depends only on the position x; this allowed us to define

$$V(x) = -\int_{x_0}^{x} F(x)dx$$

uniquely (independent of path). Such a force is denoted "**conservative**."

We can solve the energy equation formally:

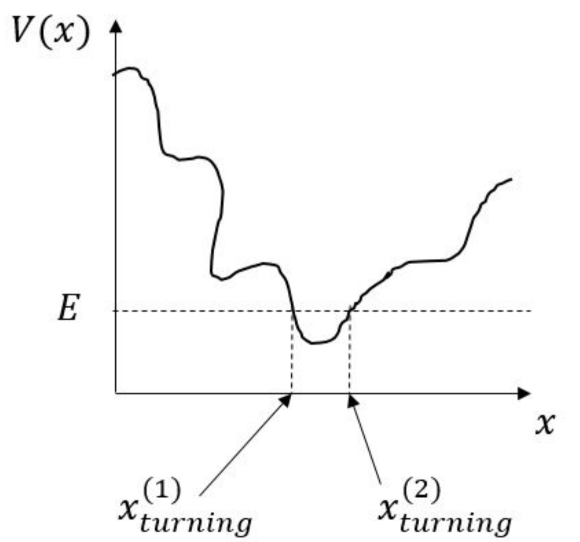
$$\frac{1}{2}mv^2 + V(x) = E \quad \Longrightarrow \quad \left(\frac{dx}{dt}\right) = \pm\sqrt{\frac{2}{m}(E - V_0)}$$

$$\Longrightarrow \boxed{\int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{2}{m}(E - V_0)}} = (t - t_0)}$$

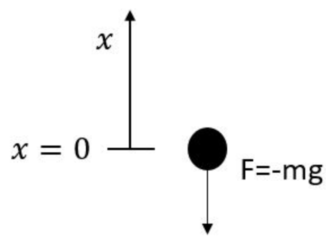
(i) For $t > t_0$, $x > x_0$ pick + (equivalent to $v > 0$)
(ii) For $t > t_0$, $x < x_0$ pick - (equivalent to $v < 0$)

This gives t as a function of x (the sign $\pm$ is determined by the initial velocity).

Note that v is only real for $V(x) \leq E$. Physically, this means that motion is restricted to values x in the region $V(x) \leq E$. Furthermore, v changes sign at the turning points satisfying $V(x_{turning}) = E$.



Example 1: Free Fall



If we assume $V(x) = 0$ at $x = 0$ we have $V(x) = +mgx$.

The initial condition

$$\begin{cases} x(t=0) = 0 \\ v(t=0) = v_0 \end{cases}$$

gives $E = \frac{1}{2}mv_0^2$

$$\frac{1}{2}mv^2 + mgx = \frac{1}{2}mv_0^2 \implies \boxed{v^2 = v_0^2 - 2gx}$$

The turning point is the point at which $v = 0$. This is just the maximum height:

$$\boxed{x_{turning} = h_{max} = \frac{v_0^2}{2g}}$$

Let's solve for time t as a function of x: $\quad v = \pm\sqrt{v_0^2 - 2gx}$. We pick the sign $+$ because the initial condition is $v_0 > 0$.

$$\int_0^x \frac{dx}{\pm\sqrt{v_0^2 - 2gx}} = t \implies -\frac{1}{g}\sqrt{v_0^2 - 2gx}\,\Big|_0^x = t$$

$$\implies -\frac{1}{g}(\sqrt{v_0^2 - 2gx} - v_0) = t$$

$$\implies \sqrt{v_0^2 - 2gx} = v_0 - gt \quad \text{or} \quad (v = v_0 - gt \;\text{ as before})$$

$$v_0^2 - 2gx = v_0^2 - 2gtv_0 + g^2t^2$$

$$\boxed{x = v_0 t - \frac{1}{2}gt^2}$$

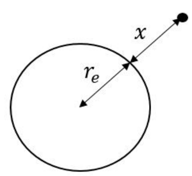
Example 2: Variation of gravity with height

Law of gravity: $F = -\dfrac{GMm}{r^2}$

Define $x = r - r_e$ where r_e is the radius of the earth. Then...

$$F(x) = -GMm\frac{1}{(x+r_e)^2} = -\frac{GMm}{r_e^2}\frac{r_e^2}{(x+r_e)^2} \implies \boxed{F(x) = -mg\frac{r_e^2}{(x+r_e)^2}}$$

since we know that $-\dfrac{GMm}{r_e^2} = -mg$ is the force of gravity as the surface of the earth.



$$F(x) = -\frac{dV}{dx} \implies \boxed{V(x) = -\frac{mgr_e^2}{(x+r_e)}}$$

Note that when $r = r_e$ (or equivalently $x = 0$) we have that $V = -mgr_e$. Now suppose that a body is launched upwards with velocity v_0:

$$E = \frac{1}{2}mv^2 - mg\frac{r_e^2}{(x+r_e)}$$

and plugging in $t = 0$, $x = 0$, and $v = v_0$ yields

$$E = \frac{1}{2}mv_0^2 - mgr_e$$

$$\implies v^2 - 2g\frac{r_e^2}{(x+r_e)} = v_0^2 - 2gr_e$$

$$v^2 = v_0^2 - 2gr_e\left(1 - \frac{r_e}{x+r_e}\right)$$

$$\boxed{v^2 = v_0^2 - 2gx\left(\frac{1}{1+\frac{x}{r_e}}\right)}$$

Note: For $x << r_e$, this reduces to the constant g cases.

The turning point is given by setting $v = 0$ and $x = h_{max}$:

$$2gh_{max}\left(\frac{1}{1+\frac{h_{max}}{r_e}}\right) = v_0^2 \implies 2gh_{max} = v_0^2 + \frac{v_0^2}{r_e}h_{max}$$

$$h_{max} = \frac{v_0^2}{2g - \frac{v_0^2}{r_e}} \implies \boxed{h_{max} = \frac{v_0^2}{2g}\left(\frac{1}{1-\frac{v_0^2}{2gr_e}}\right)}$$

Again, when ${v_0}^2 << 2gr_e$, we get the previous result.

The escape velocity v_e is the smallest v_0 such that $h_{max} = \infty$. We find this by setting the denominator equal to zero,

$$1 - \frac{v_e^2}{2gr_e} = 0 \implies v_e = \sqrt{2gr_e} \approx \boxed{\text{11 km/s for the earth}}$$

Average speed of O_2?

$\frac{1}{2}m_{O_2}\bar{v}_{O_2}^2 = \frac{1}{2}k_bT \implies \bar{v}_{O_2} = \sqrt{\frac{k_bT}{m_{O_2}}}$

$m_{O_2} = 2 \times 16 \times 1.66 \times 10^{-27}kg = 5.3 \times 10^{-26}kg$

$k_bT = 1.38 \times 10^{-23}J/K \times 300K = 4.1 \times 10^{-21}J$

$$\implies \quad \boxed{\bar{v}_{O_2} = \sqrt{\frac{4 \times 10^{-21}}{5.3 \times 10^{-26}}} = 0.3km/s << v_{escape}}.$$

Now since $v \propto \sqrt{m}$ we know that

$$\bar{v}_{H_2} = \sqrt{\frac{m_{O_2}}{m_{H_2}}}\bar{v}_{O_2} = \sqrt{16}\bar{v}_{O_2} = 1.2km/s << v_{escape}$$

Why don't we have H_2 in our atmosphere? At $T = 3 \times 10^4K$ (millions of years ago...) we have

$$\bar{v}_{O_2} = 3km/s < v_{escape}$$
$$\bar{v}_{H_2} = 12km/s > v_{escape}$$