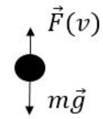
Lecture 3: Velocity Dependent Forces

The viscous force that a fluid exerts on a particle depends on velocity, $F = F(v)$. For most problems, it suffices to expand F in powers of v and keep only the 1^{st} and 2^{nd} orders:

$$F(v) = -c_1 v - c_2 v|v| - c_3 v^3 - ...$$

The modulus appears because F should be **against** v. In other words, $\text{sgn}(F) = -\text{sgn}(v)$



For a sphere in air:

$$\begin{cases} c_1 \approx 1.55 \times 10^{-4} \left(\frac{D}{1m}\right) \frac{Ns}{m} & \text{(D is diameter of sphere)} \\ c_2 \approx 0.22 \left(\frac{D}{1m}\right)^2 \frac{Ns^2}{m^2} \end{cases}$$

Note: These coefficients depend on shape. $c_1 \propto \sqrt{A}$, $\quad c_2 \propto A_{crosssection}$. Which dominates? Depends on D and v.

$$\text{ratio} = \frac{F_{quad}}{F_{linear}} = \frac{c_2 v^2}{c_1 v} = \frac{0.22}{1.55 \times 10^{-4}} Dv \quad = \quad \frac{1.4 \times 10^3}{m^2/s} Dv$$

Hence when $v << \frac{1}{1.4 \times 10^3 D} \frac{m^2}{s}$, c_1 dominates!

Baseball: $D = 0.07m \implies$ Quad dominates (ratio > 1) when

$$(1.4 \times 10^3) Dv > 1 \implies v > (1.4 \times 10^3 \times 7 \times 10^{-2})^{-1}$$
$$v > 10^{-2} m/s = 1cm/s$$

Basketball: $D = 0.25m \implies$ ratio > 1 when $v > 3.6cm/s$.

Horizontal Motion with Linear Resistance

$$-c_1 v = m\frac{dv}{dt} \implies \int dt = -\frac{m}{c_1}\int_{v_0}^{v}\frac{dv}{v} \implies t = -\frac{m}{c_1}ln\left(\frac{v}{v_0}\right)$$

$$\implies v = v_0 e^{-\frac{c_1}{m}t}$$

Since $v = dx/dt$ it follows that

$$\int_0^x dx = v_0\int_0^t dt \cdot e^{-\frac{c_1}{m}t} \implies x = v_0\left(-\frac{m}{c_1}\right)\left(e^{-\frac{c_1}{m}t} - 1\right)$$

$$\boxed{x(t) = \left(\frac{v_0 m}{c_1}\right)\left(1 - e^{-\frac{c_1}{m}t}\right)}$$

Note that as $t \to \infty$ we have that $x \to v_0 m/c_1$.

Quadratic Resistance

$$-c_2 v^2 = m\frac{dv}{dt} \quad \text{(assumed } v > 0\text{)}$$

$$t = -\frac{m}{c_2}\int_{v_0}^{v}\frac{dv}{v^2} \;=\; \frac{m}{c_2}\cdot\frac{1}{v}\Big|_{v_0}^{v} \;=\; \frac{m}{c_2}\left(\frac{1}{v} - \frac{1}{v_0}\right)$$

So it follows that

$$\left(\frac{c_2}{m}\right)t + \frac{1}{v_0} = \frac{1}{v} \implies \boxed{v = \frac{v_0}{1 + \left(\frac{v_0 c_2}{m}\right)t}}$$

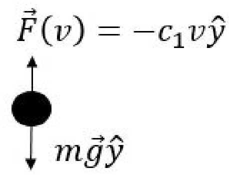
From this point onwards we will define $k = v_0 c_2/m$. Note that $v \sim 1/t$ when $t \to \infty$.

$$\int_0^x dx = v_0\int_0^t \frac{dt}{1 + kt} = \frac{v_0}{k}ln(1 + kt)$$

$$\boxed{x(t) = \frac{m}{c_2}\ln(1 + kt)}$$

In this case, $x \to \infty$ when $t \to \infty$. The v^2 force gets weaker and weaker as the particle reduces velocity. Obviously, for v very small, we need to consider the linear term.

Vertical Fall and Vertical Velocity



For a downward fall we have $v < 0$ so $F(v) = -c_1 v\hat{y}$ is directed upwards.

$$-mg - c_1 v = m\frac{dv}{dt} \implies dt = -\frac{mdv}{mg + c_1 v}$$

$$\implies t = -\left(\frac{m}{c_1}\right)\ln(mg + c_1 v)\Big|_{v_0}^{v}$$

$$\implies t = -\left(\frac{m}{c_1}\right)\ln\left(\frac{mg + c_1 v}{mg + c_1 v_0}\right)$$

$$(mg + c_1 v_0)e^{-\frac{c_1}{m}t} = (mg + c_1 v) \implies \boxed{v(t) = \left(\frac{mg}{c_1} + v_0\right)e^{-\frac{c_1}{m}t} - \left(\frac{mg}{c_1}\right)}$$

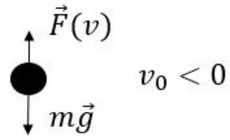
When $t \to \infty$, v approaches the terminal velocity $v_t = -mg/c_1$. (Note that $dv/dt = 0$ for $v = v_t$! The drag force has canceled the weight). The characteristic time for this approach is $\tau = m/c_1$.

$$\boxed{v(t) = v_t(1 - e^{-\frac{t}{\tau}}) + v_t(1 - e^{-\frac{t}{\tau}})}$$

We regard $v_t(1 - e^{-\frac{t}{\tau}})$ as the "fade in" and $v_t(1 - e^{-\frac{t}{\tau}})$ as the "fade out."

For $v_0 = 0$: v is within 1% of v_t after $t = 5\tau$.

Quadratic Case



$$m\frac{dv}{dt} = -mg + c_2 v^2$$

The equation above is valid for $v_0 < 0$. Otherwise, we'd need to be careful with the sign!

$$\implies m\frac{dv}{dt} = -mg\left(1 - \frac{c_2}{mg}v^2\right)$$

$$\frac{dv}{dt} = -g\left(1 - \frac{v^2}{v_t^2}\right)$$

where v_t is the terminal velocity:

$$\boxed{v_t = -\sqrt{\frac{mg}{c_2}}}$$

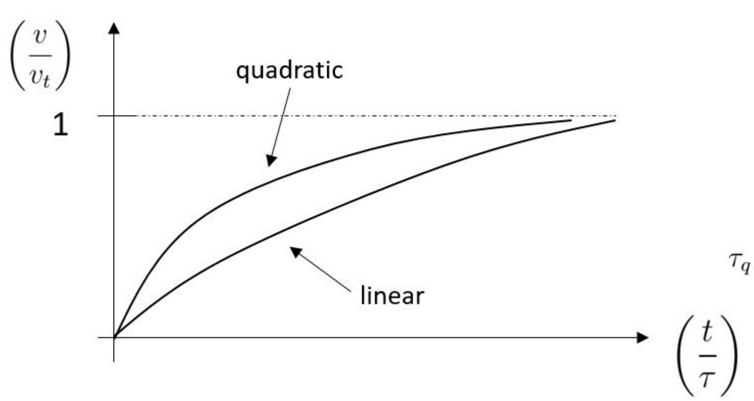
Now we have that

$$\int_{v_0}^{v} \frac{dv}{1 - \frac{v^2}{v_t^2}} = -gt$$

$$v_t \tanh^{-1}\left(\frac{v}{v_t}\right)\Bigg|_{v_0}^{v} = -gt \implies \tanh^{-1}\left(\frac{v}{v_t}\right) = -\frac{gt}{v_t} + \tanh^{-1}\frac{v_0}{v_t}$$

$$\implies \boxed{v(t) = v_t \tanh\left(\frac{t}{\tau} + \tanh^{-1}\left(\frac{v_0}{v_t}\right)\right)}$$

where τ is defined as $\boxed{\tau = -v_t/g} > 0$. Note that $v(t = 5\tau) = 0.99991 v_t$ for $v_0 = 0$. This is faster than linear!



$$\tau_l = \frac{m}{c_1} \quad , \quad |v_{tl}| = \frac{mg}{c_1}$$

$$\tau_q = \sqrt{\frac{m}{c_2 g}} \quad , \quad |v_{tq}| = \sqrt{\frac{mg}{c_2}}$$

Characteristic Distance for Approaching Terminal Velocity?

Question: Does it depend on v_0? We use the fact that

$$\frac{dv}{dt} = \frac{1}{2}\frac{d}{dy}(v^2) \qquad \left(= \frac{dy}{dt}\frac{dv}{dy}\right)$$

(Note that $y < 0$ because the particle is going down)

$$\implies \frac{d}{dy}(v^2) = -2g\left(1 - \frac{v^2}{v_t^2}\right)$$

We now define

$$u = \left(1 - \frac{v^2}{v_t^2}\right); \qquad \frac{du}{dy} = -\frac{1}{v_t^2}\frac{d(v^2)}{dy} = +\frac{2g}{v_t^2}u$$

$$\implies u(y) = u(y=0)e^{+\frac{2g}{v_t^2}y}$$

$$u(y) = \left(1 - \frac{v_0^2}{v_t^2}\right)e^{+\frac{y}{y_0}}$$

where we have that

$$y_0 = \frac{v_t^2}{2g} = \left(\sqrt{\frac{mg}{c_2}}\right)^2\frac{1}{2g} = \frac{m}{2c_2}$$

Hence the characteristic length is **independent** of both v_0 and g.

$$u = \left(1 - \left(\frac{v}{v_t}\right)^2\right) \implies \left(1 - \left(\frac{v}{v_t}\right)^2\right) = \left(1 - \left(\frac{v_0}{v_t}\right)^2\right)e^{+\frac{y}{y_0}}$$

$$\implies \boxed{v^2 = v_t^2(1 - e^{+\frac{y}{y_0}}) + v_0^2 e^{+\frac{y}{y_0}}}$$

Can also show: For Linear Drag: $y_0 = v_t^2/g$ (no factor of 2)

$\boxed{\underline{\text{Raindrop:}}\ \text{Linear}, \quad v_t = 0.33m/s \quad , \quad y_0 = 1cm \quad , \quad \tau = 0.034s}$

$\boxed{\underline{\text{Basketball:}}\ \text{Quadratic}, \quad v_t = 20.6m/s \quad , \quad y_0 = 21.6m \quad , \quad \tau = 2.1s}$