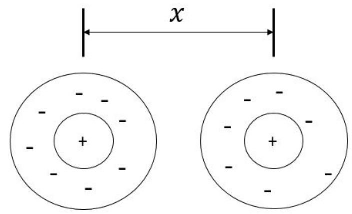
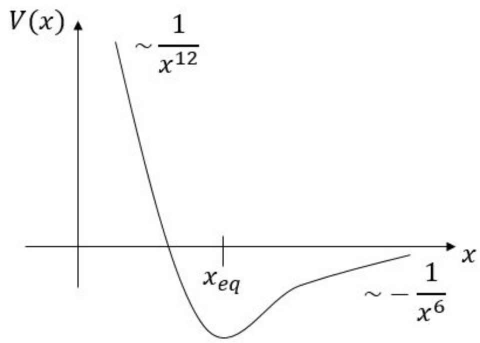
Lecture 4: Harmonic Motion



Two mutual atoms interacting:

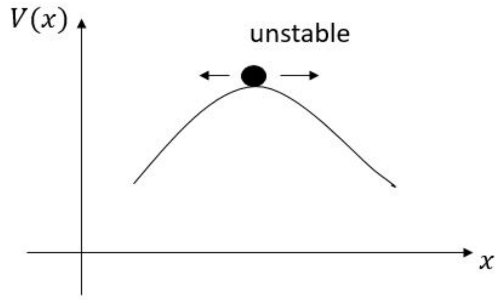


$$V(x) = \frac{A}{x^{12}} - \frac{B}{x^6}$$

To find the equilibrium point, where the force is zero:

$$F = -\frac{dV(x)}{dx} = \frac{12A}{x^{13}} - \frac{6B}{x^7} = 0 \quad \Longrightarrow \quad \frac{12A}{x^{13}} = \frac{6B}{x^7} \quad \Longrightarrow \quad x_{eq} = \sqrt[6]{\frac{2A}{B}}$$

Stable or unstable equilibrium? $\boxed{\text{Stable: } \frac{d^2V}{dx^2} > 0}$ $\boxed{\text{Unstable: } \frac{d^2V}{dx^2} < 0}$



To determine stability, we expand $V(x)$ about x_{eq}.

$$V(x) \approx V(x_{eq}) + \frac{dV}{dx} \cdot (x - x_{eq}) + \frac{1}{2}\frac{d^2V}{dx^2} \cdot (x - x_{eq})^2 + \sigma(x - x_{eq})^3 + ...$$

where all derivatives are evaluated at $x = x_{eq}$. We know that $dV/dx = 0$. For the second order derivative:

$$\frac{d^2V}{dx^2} \equiv k = -\frac{13 \cdot 12A}{x_{eq}^{14}} + \frac{7 \cdot 6B}{x_{eq}^{8}}$$

$$\implies \quad \boxed{V(x) = \text{const.} + \frac{1}{2}k(x - x_{eq})^2}$$

In general, for a number of physical problems involving equilibrium, we can always expand in powers of deviations from equilibrium. $\implies$ Minimum "Linear" model.

Later, we will show that terms like $G(x - x_{eq})^3$ and $G(x - x_{eq})^4$ give us non-linear effects and chaos.

The associated force is

$$F(x) = -\frac{dV}{dx} = -k(x - x_{eq}) \qquad \text{("Hooke's law for springs")}$$

Newton's law: (drop x_{eq} for simplicity):

$$m\ddot{x} = -kx \quad \implies \quad \boxed{\ddot{x} + \frac{k}{m}x = 0}$$

To find the solution, we plug in

$$x(t) = A\sin(w_0 t + \phi_0) \qquad \qquad \dot{x}(t) = Aw_0\cos(w_0 t + \phi_0)$$

And hence, by Newton's law, we have that

$$-Aw_0^2\sin(w_0 t + \phi_0) + \left(\frac{k}{m}\right)A\sin(w_0 t + \phi_0) = 0$$

$$\implies \quad \left(\frac{k}{m} - w_0^2\right)\sin(w_0 t + \phi_0) = 0 \quad \implies \quad \boxed{w_0 = \sqrt{\frac{k}{m}}}$$

To find the parameters A and ϕ_0, we relate to the initial conditions $x(t = 0) = x_0$ and $v(t = 0) = v_0$.

$$\implies \quad \begin{cases} x(t=0) = a\sin(\phi_0) = x_0 \\ \\ \dot{x}(t=0) = Aw_0\cos(\phi_0) = v_0 \end{cases}$$

$$\implies \quad \begin{cases} \frac{1}{w_0}\tan(\phi_0) = \frac{x_0}{v_0} \implies \boxed{\phi_0 = \tan^{-1}\left(\dfrac{w_0 x_0}{v_0}\right)} \\ \\ A^2\sin^2(\phi_0) + A^2\cos^2(\phi_0) = x_0^2 + \left(\frac{v_0}{w_0}\right)^2 \implies \boxed{A = \sqrt{x_0^2 + \left(\dfrac{v_0}{w_0}\right)^2}} \end{cases}$$

Period of Motion: $T_0 = \dfrac{2\pi}{w_0}$

Superposition Principle:

$$\text{if} \qquad \begin{cases} x_1(t) = A_1\sin(w_0 t + \phi_1) \\ x_2(t) = A_2\sin(w_0 t + \phi_2) \end{cases}$$

are solutions, then $x_1(t) + x_2(t)$ is also a solution. This happens because the differential equation is linear (no x^2, $\dot{x}^2$, etc...)

Hertz or rad/s?

Angular Frequency: $w_0 = \dfrac{2\pi}{T_0} \implies$ units are rad/s

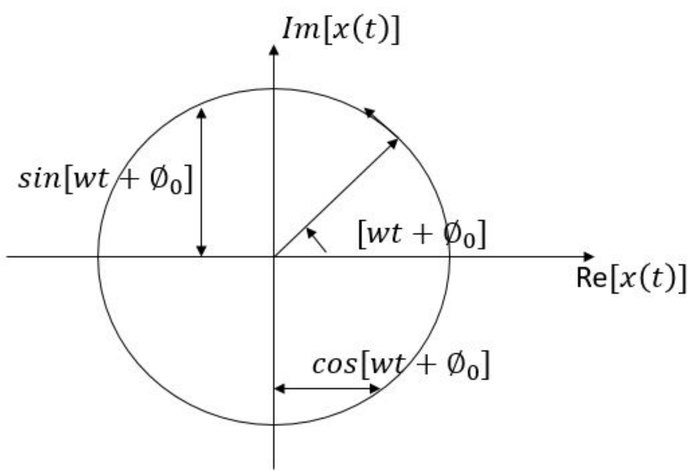
Frequency: $f = \dfrac{1}{T_0} = \dfrac{w_0}{2\pi} \implies$ units are Hertz (s^{-1}), or cycles per second.

Complex Solutions

$$x(t) = Ae^{i(w_0 t+\phi_0)} = A[\cos(w_0 t + \phi_0) + i\sin(w_0 t + \phi_0)]$$

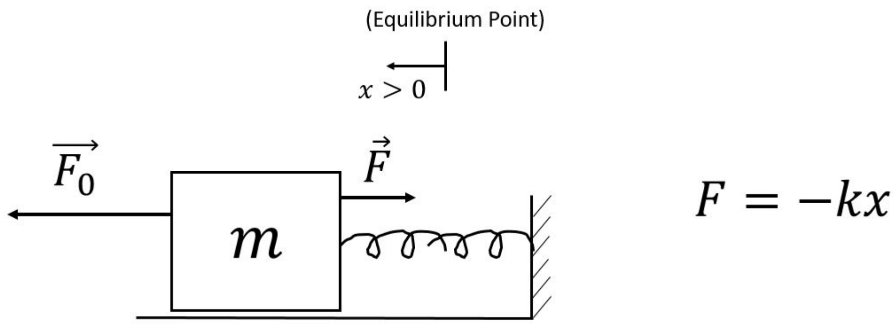
$$\dot{x}(t) = iw_0 x(t)$$

$$\ddot{x}(t) = (iw_0)^2 x(t) \implies \ddot{x} = -w_0^2 x = \boxed{-\frac{k}{m}x}$$



Interpretation: Particle rotating in a circle with constant speed.

Example 1: Effect of a constant force on a spring

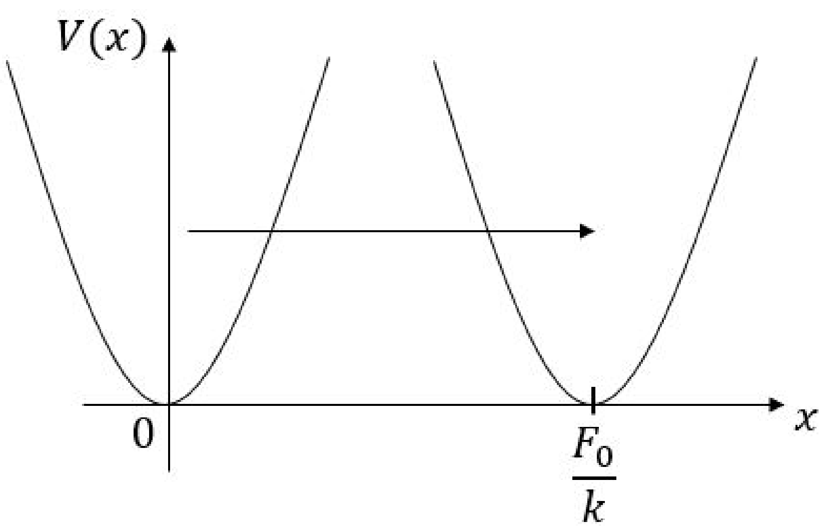


$$m\ddot{x} \;=\; -kx + F_0 \;=\; -k\left(x - \frac{F_0}{k}\right) \;=\; -kx'$$

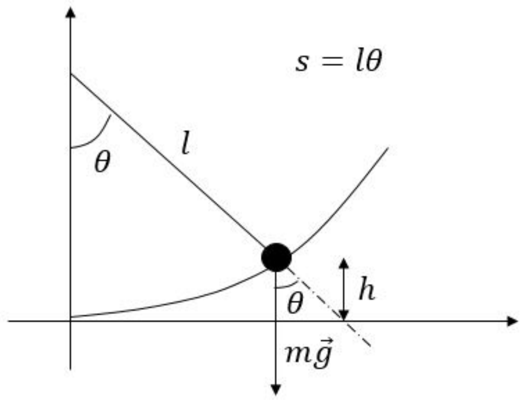
where we define $x' = x - F_0/k$ and where F_0 is a constant force. Thus

$$\ddot{x}' = \ddot{\left(x - \frac{F_0}{k}\right)} = \ddot{x} \implies \boxed{m\ddot{x}' = -kx'}$$

Any constant force applied to a harmonic oscillator merely shifts the equilibrium position. This implies that the equation of motion is unchanged if we shift the displacement variables.



Example 2: Simple Pendulum



We have that

$$m\ddot{s} = -mg\sin(\theta)$$

$$ml\ddot{\theta} = -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^3}{5!} - ...\right) \approx -mg\theta \quad \Longrightarrow \quad \boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

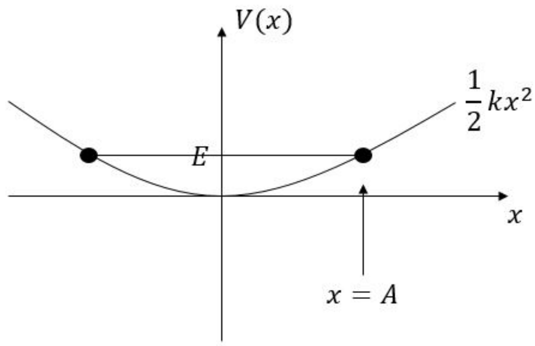
Hence the frequency of oscillations is $w_0 = \sqrt{g/l}$ (independent of amplitude). We can use this fact for neat applications, including how to build a clock.

Suppose we want the period of a clock to be $T_0 = 2s$. How long should we make the pendulum?

$$T_0 = 2\pi\sqrt{\frac{l}{g}} \quad \Longrightarrow \quad l = g\left(\frac{T_0}{2\pi}\right)^2 = 0.9936m$$

Energy of a Harmonic Oscillator

$$\begin{cases} x(t) = A\sin(w_0 t + \phi_0) \\ \dot{x}(t) = Aw_0 \cos(w_0 t + \phi_0) \end{cases}$$



x is reaches its maximum value when $\dot{x} = 0$, $\quad x_{max} = \pm A$ are turning points.

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}mA^2w_0^2\cos^2(w_0 t + \phi_0) + \frac{1}{2}kA^2\sin^2(w_0 t + \phi_0)$$

$$\boxed{E = \frac{1}{2}kA^2}$$

We can also show that

$$< K > = < \frac{1}{2}m\dot{x}^2 > = \frac{1}{T_0}\int_0^{T_0} dt \frac{1}{2}m\dot{x}^2 = \frac{1}{4}kA^2$$

$$< V > = < \frac{1}{2}mx^2 > = \frac{1}{T_0}\int_0^{T_0} dt \frac{1}{2}kx^2 = \frac{1}{4}kA^2$$

$$< K > = < V > = \frac{E}{2}$$

We call this the "swing" energy between $< K >$ and $< V >$