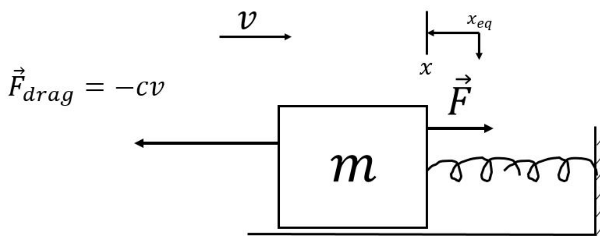
Lecture 5: Damped Harmonic Motion



$x > 0$

$\underline{v < 0} : F_{damping} > 0 \quad , \quad F_{rest} < 0$

$\underline{v > 0} : F_{damping} < 0 \quad , \quad F_{rest} < 0$

$$m\ddot{x} = -kx - cv \quad \Longrightarrow \quad \ddot{x} + \left(\frac{k}{m}\right)x + \left(\frac{c}{m}\right)\dot{x} = 0$$

We now define $w_0^2 = k/m$ and $2\gamma = c/m$. Our equation becomes

$$\boxed{\ddot{x} + 2\gamma\dot{x} + w_0^2 x = 0}$$

To solve this ODE, we use the differential operator method. We write the equation as

$$D\left(\frac{d}{dt}, \frac{d^2}{dt^2}\right)x = 0$$

$$D\left(\frac{d}{dt}, \frac{d^2}{dt^2}\right) = \left(\frac{d}{dt} + 2\gamma\frac{d^2}{dt^2} + w_0^2\right) \quad = \quad \left(\frac{d}{dt} - r_+\right)\left(\frac{d}{dt} - r_-\right)$$

We have a quadratic equation d/dt. We find the roots:

$$r_\pm = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4w_0^2}}{2} \quad = \quad \left(-\gamma \pm \sqrt{\gamma^2 - w_0^2}\right)$$

Hence it follows that

$$D\left(\frac{d}{dt}, \frac{d^2}{dt^2}\right) = \left(\frac{d}{dt} + \gamma - \sqrt{\gamma^2 - w_0^2}\right)\left(\frac{d}{dt} + \gamma + \sqrt{\gamma^2 - w_0^2}\right) \quad = \quad D_1\left(\frac{d}{dt}\right) \cdot D_2\left(\frac{d}{dt}\right)$$

Let $\boxed{q = \sqrt{\gamma^2 - w_0^2}}$

Our differential equation becomes

$$\left(\frac{d}{dt}+\gamma-q\right)\left(\frac{d}{dt}+\gamma+q\right)x(t)=0$$

Note that D_1 and D_2 are **commutative** operators. In other words, $D_1D_2 = D_2D_1$. Therefore there are two possible solutions:

$$\begin{cases} x_1(t)=e^{-(\gamma-q)t} \quad \text{since} \quad D_1x_1(t)=0 \\ \\ x_2(t)=e^{-(\gamma+q)t} \quad \text{since} \quad D_2x_2(t)=0 \end{cases}$$

$$\Longrightarrow \quad \boxed{x(t)=A_1e^{-(\gamma-q)t}+A_2e^{-(\gamma+q)t}}$$

You can check this solution directly by substitution into the second order differential equation. Remember that $q=\sqrt{\gamma^2-w_0^2}$.

Three Possible Scenarios:

(i) q real > 0 OVERDAMPING

(ii) $q = 0$ CRITICAL DAMPING

(iii) q imaginary UNDERDAMPING

Case 1: Overdamping

This occurs when $\gamma > w_0$.

$$x(t) = A_1 e^{-(\gamma - q)t} + A_2 e^{-(\gamma + q)t}$$

The A_1 term decays slowly whereas the A_2 term decays fast. Now consider

$$x(t=0) = x_0 \implies A_1 + A_2 = x_0$$
$$\dot{x}(t=0) = 0 \implies -(\gamma - q)A_1 - (\gamma + q)A_2 = 0 \implies \frac{A_1}{A_2} = -\frac{\gamma + q}{\gamma - q}$$

Now assume $\gamma >> w_0 \implies q = \sqrt{\gamma^2 - w_0^2} = \gamma\sqrt{1 - (\frac{w_0}{\gamma})^2} \approx \gamma[1 - \frac{1}{2}(\frac{w_0}{\gamma})^2]$

$$\implies \frac{A_1}{A_2} = -\frac{2\gamma}{\frac{1}{2}\left(\frac{w_0}{\gamma}\right)^2 \gamma} = -4\left(\frac{\gamma}{w_0}\right)^2 \implies \boxed{A_1 >> A_2}$$

From this, it follows that

$$x(t) \approx x_0 e^{-(\gamma - q)t} \approx x_0 e^{-\frac{1}{2}\left(\frac{w_0}{\gamma}\right)^2 \gamma t}$$

This solution is valid at longer times, because we dropped A_2. Note that $\dot{x}$ is approximately less than or equal to zero. For simplicity, we write

$$x(t) \approx x_0 e^{-t/\tau} \quad \text{where} \quad \tau = 2\left(\frac{\gamma}{w_0}\right)^2 \frac{1}{\gamma} >> \frac{1}{\gamma}$$

In this situation, it takes a long time to reach equilibrium, despite being overdamped!

Case 2: Critical Damping

This occurs when $\gamma = w_0 \quad \Longrightarrow \quad q = 0$.

The solution $x_1(t)$ and $x_2(t)$ are no longer independent, because the functions are the same ($x_1(t) = x_2(t) = e^{-\gamma t}$). Our method of swapping the order of the operators to find two independent solutions no longer works. We need to find an additional solution.

$$\left(\frac{d}{dt} + \gamma\right)^2 x(t) = 0$$

We define

$$u(t) = \left(\frac{d}{dt} + \gamma\right) x(t) \quad \Longrightarrow \quad \left(\frac{d}{dt} + \gamma\right) u(t) \;=\; 0 \quad \Longrightarrow \quad u(t) = Ae^{-\gamma t}$$

$$\Longrightarrow \quad A = e^{\gamma t}\left(\frac{d}{dt} + \gamma\right) x(t) \;=\; \frac{d}{dt}\left(x(t)e^{\gamma t}\right)$$

$$\Longrightarrow \quad \int_0^t A dt = \int_0^t d(xe^{\gamma t}) \quad \Longrightarrow \quad At = x(t)e^{\gamma t} - x_0$$

$$\Longrightarrow \quad \boxed{x(t) = (At + x_0)e^{-\gamma t}}$$

Also from $\dot{x}(t = 0) = v_0$ we get that $A = (v_0 + \gamma x_0)$.

Critical damping is desired in many applications, such as vehicle suspensions. It is the fastest way to reach equilibrium **without** oscillating back and forth.

Proof: (for $\dot{x}(t = 0) = 0$)

$$x(t) = x_0(\gamma t + 1)e^{-\gamma t} \sim x_0 \gamma t e^{-\gamma t} \qquad \text{(assymptotes)}$$
$$x_{overdamped}(t) \sim x_0 e^{(-\gamma - q)t}$$

$$\Longrightarrow \quad \text{ratio} = \gamma t e^{-qt}$$

and as $t \to \infty$, the ratio approaches zero.

Case 3: Underdamping

This occurs when $\gamma < w_0 \quad \Longrightarrow \quad q = i\sqrt{w_0^2 - \gamma^2} \; = \; iw_d$ where w_d is the damped oscillation frequency. It is always the case that $w_d < w_0$.

$$\begin{aligned} x(t) &= c_+ e^{-(\gamma - iw_d)t} + c_- e^{-(\gamma + iw_d)t} \\ &= e^{-\gamma t}\left(c_+ e^{iw_d t} + c_- e^{-iw_d t}\right) \end{aligned}$$

Note that x must be real. Hence $c_+ e^{iw_d t} = \left(c_- e^{-iw_d t}\right)^* \quad \Longrightarrow \quad c_+ = c_-{}^* = c^*$.

$$x(t) = e^{-\gamma t}\left(c^* e^{iw_d t} + c e^{-iw_d t}\right)$$

For convenience, we write $c = i(A/2)e^{-i\phi_0}$

$$\begin{aligned} x(t) &= e^{-\gamma t}\left(-i\frac{A}{2}e^{iw_d t + \phi_0} + i\frac{A}{2}e^{-iw_d t + \phi_0}\right) \\ &= Ae^{-\gamma t}\left(\frac{e^{iw_d t + \phi_0} - e^{-iw_d t + \phi_0}}{2i}\right) \end{aligned}$$

$$\boxed{x(t) = Ae^{-\gamma t}\sin(w_d t + \phi_0)}$$

With a longer period due to damping, $T_d > T_0$ ($w_d < w_0$).