# Lecture 6: Damped Harmonic Motion: Energy, Quality Factor, Phase Space

**Review from last lecture:**

$$m\ddot{x} = -kx - cv \quad \Longrightarrow \quad \boxed{\ddot{x} + 2\gamma\dot{x} + w_0^2 x = 0}$$

$$\gamma = \frac{c}{2m} \quad , \quad w_0 = \sqrt{\frac{k}{m}}$$

**(i) Overdamping**: $\boxed{\gamma > w_0}$

$$x(t) = A_1 e^{-(\gamma - q)t} + A_2 e^{-(\gamma + q)t} \qquad q = \sqrt{\gamma^2 - w_0^2}$$

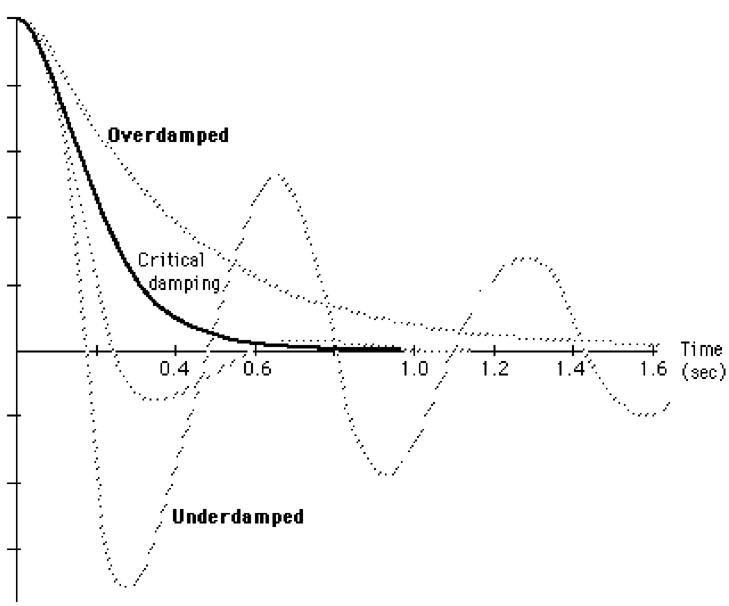
**(ii) Critical Damping**: $\boxed{\gamma = w_0}$

$$x(t) = ((v_0 + \gamma x_0)t + x_0)e^{-\gamma t}$$

**(iii) Underdamping**: $\boxed{\gamma < w_0}$

$$x(t) = Ae^{-\gamma t}\sin(w_d t + \phi_0) \qquad \boxed{w_d = \sqrt{w_0^2 - \gamma^2}} < w_0$$

For example:



## Energy Loss

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} \;\;=\;\; (m\ddot{x} + kx)\dot{x} = -c\dot{x}^2 < 0$$

Hence $dE/dt < 0$ **always** and $E \to 0$. We also have

$$(m\ddot{x} + kx)\dot{x} = F_{damping}\dot{x} \quad (= Fv)$$

Recall that Power$= Fv$.

## Quality Factor

For a weakly damped system, $Q >> 1$. For a strongly damped system, $Q \sim 1$ or $Q << 1$. $Q$ is defined as

$$Q \equiv 2\pi \frac{E}{|\Delta E|}$$

We can think of $E/|\Delta E|$ as 1/fraction of energy lost per cycle. (This concept only applies to underdamped systems!)

$$\Delta E = \int_0^{T_d} \dot{E} dt = -c\int_0^{T_d} dt\dot{x}^2$$

We will use

$$x(t) = Ae^{-\gamma t}\sin(w_d t) \quad \Longrightarrow \quad \dot{x}(t) = Aw_d e^{-\gamma t}\cos(w_d t) - A\gamma e^{-\gamma t}\sin(w_d t)$$

Hence we have

$$\Delta E = -c\int_0^{T_d} dt[A^2{w_d}^2 e^{-2\gamma t}\cos^2(w_d t) - 2A^2 w_d \gamma e^{-2\gamma t}\sin(w_d t)\cos(w_d t) + A^2\gamma^2 e^{-2\gamma t}\sin^2(w_d t)]$$

Now let $\theta = w_d t \implies dt = d\theta/w_d$.

$$\Delta E \approx -\frac{cA^2}{w_d}e^{-2\gamma t}\int_0^{2\pi} d\theta[w_d^2\cos^2(\theta) - 2w_d\gamma\sin(\theta)\cos(\theta) + \gamma^2\sin^2(\theta)]$$

Recall that $\cos^2\theta = (1+\cos 2\theta)/2$ and $\sin^2\theta = (1-\cos 2\theta)/2$. Hence

$$\Delta E \approx -\frac{cA^2}{w_d}\frac{[w_d^2+\gamma^2]}{2}2\pi e^{-2\gamma t} \quad = \quad -cA^2w_0^2\left(\frac{\pi}{w_d}\right)e^{-2\gamma t} \quad = \quad -m\gamma A^2w_0^2T_d e^{-2\gamma t}$$

In the line above, we use the fact that $w_d^2+\gamma^2 = w_0^2$ and that $c = 2m\gamma$. If we write $2\gamma = 1/\tau$ we have

$$\Delta E = -\left(\frac{1}{2}mw_0^2A^2e^{-t/\tau}\right)\left(\frac{T_d}{\tau}\right) \quad \Longrightarrow \quad \boxed{\frac{|\Delta E|}{E} = \frac{T_d}{\tau}}$$

$$Q = 2\pi\frac{E}{|\Delta E|} = \frac{2\pi\tau}{T_d} = \frac{2\pi\tau}{2\pi/w_d} = w_d\tau \qquad \left(= \frac{w_d}{2\gamma}\right)$$

Note that min Q= 0 because $w_d = \sqrt{w_0^2-\gamma^2}$

| Event | Q |
|---|---|
| Earthquakes | 250-1400 |
| Piano String | 3000 |
| Excited Atom | $10^7$ |
| Excited $Fe^{57}$ nucleus | $10^{12}$ |

## Phase Space

Trajectory in the px plane: $(p, x)$

$p = m\dot{x}$

**Example:** Simple Harmonic Oscillator

$$x(t) = A\sin(w_0 t + \phi_0) \qquad \dot{x}(t) = Aw_0\cos(w_0 t + \phi_0)$$

$$\implies \quad \frac{x^2}{A^2} + \frac{(m\dot{x})^2}{m^2 w_0^2 A^2} \;=\; 1 \quad \implies \quad \boxed{\frac{x^2}{A^2} + \frac{p^2}{m^2 w_0^2 A^2}} \qquad \text{(ellipse)}$$

This is equivalent to energy conservation.

$$\frac{1}{2}mw_0^2 x^2 + \frac{p^2}{2m} = \frac{1}{2}mw_0^2 A^2 = E$$

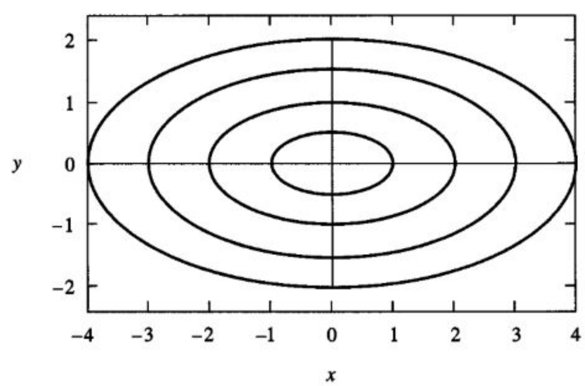
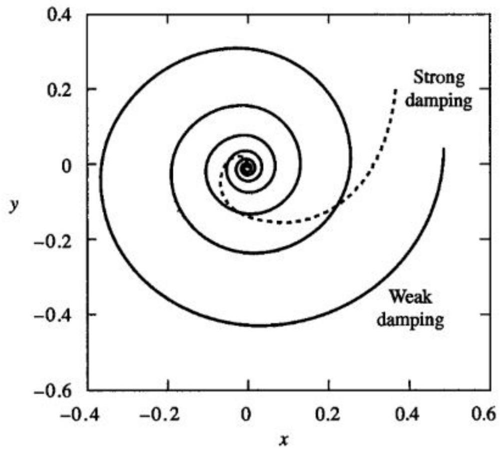



**Figure 3.5.1** Phase-space plot for the simple harmonic oscillator ($\omega_0 = 0.5\ \mathrm{s}^{-1}$). No damping force ($\gamma = 0\ \mathrm{s}^{-1}$).

Each ring represents different initial conditions, or different energies. We refer to these as *concentric ellipses.*

**Note:** Trajectories in the plane can never cross. **Proof:** Each $(x_0, p_0)$ leads to a unique solution; if two different trajectories cross, the initial condition on the crossing point will have two different solutions $\implies$ absurd.

## Underdamped




## Critically Damped

$$x(t) = [(v_0 + \gamma x_0)t + x_0]e^{-\gamma t} \quad , \quad \dot{x}(t) = (v_0 + \gamma x_0)e^{-\gamma t} - \gamma[(v_0 + \gamma x_0)t + x_0]e^{-\gamma t}$$

Note that

$$\dot{x}(t) = (v_0 + \gamma x_0)e^{-\gamma t} - \gamma x$$

and hence we have that

$$\boxed{\dot{x} + \gamma x = (v_0 + \gamma x_0)e^{-\gamma t}}$$

As t becomes large, this approaches zero. Hence $p + m\gamma x \to 0 \implies$ approaches asymptotically to the straight line $p = -(m\gamma)x$

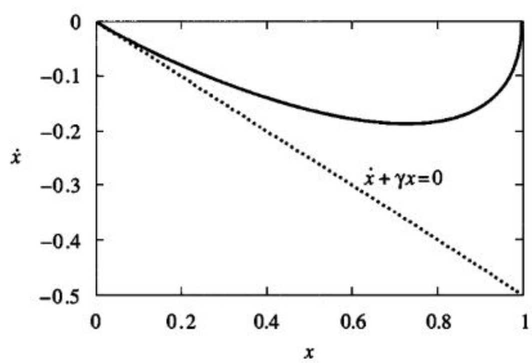



**Figure 3.5.4** Phase-space plot for the simple harmonic oscillator ($\omega_0 = 0.5\ \mathrm{s}^{-1}$). Critical damping ($\gamma = 0.5\ \mathrm{s}^{-1}$).

## Overdamped

For this part we assume $v_0 = 0$

$$x(t) = \frac{\gamma+q}{2q} x_0 e^{-(\gamma-q)t} - \frac{\gamma-q}{2q} x_0 e^{-(\gamma+q)t}$$

$$\dot{x}(t) \;=\; -(\gamma-q)\left(\frac{\gamma+q}{2q}\right) x_0 e^{-(\gamma-q)t} + (\gamma-q)\left(\frac{\gamma+q}{2q}\right) x_0 e^{-(\gamma-q)t}$$

$$= \; -(\gamma-q)\left(\left(\frac{\gamma+q}{2q}\right) x_0 e^{-(\gamma-q)t} - \left(\frac{\gamma-q}{2q}\right) x_0 e^{-(\gamma-q)t}\right) - \frac{(\gamma-q)^2}{2q} x_0 e^{-(\gamma+q)t} + \frac{\gamma^2-q^2}{2q} x_0 e^{-(\gamma+q)t}$$

These equations are rather cumbersome, but at this point, note that

$$\left(\left(\frac{\gamma+q}{2q}\right) x_0 e^{-(\gamma-q)t} - \left(\frac{\gamma-q}{2q}\right) x_0 e^{-(\gamma-q)t}\right) = x$$

and thus we conclude that

$$\dot{x} + (\gamma-q)x = (\gamma-q)x_0 e^{-(\gamma+q)t} \to 0$$
$$p + m(\gamma-q)x \to 0$$

**Figure 3.5.5** Phase-space plot for the simple harmonic oscillator ($\omega_0 = 0.5\ s^{-1}$). Overdamping ($\gamma = 1\ s^{-1}$).