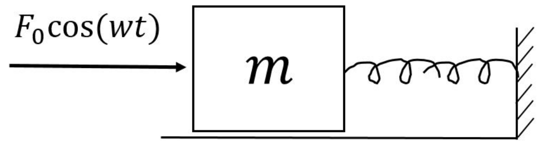
Lecture 7: Forced Harmonic Motion and Resonance



We have

$$m\ddot{x} = -kx - c\dot{x} + F_0\cos(\omega t)$$

$$\boxed{\ddot{x} + 2\gamma\dot{x} + w_0^2 x = \left(\frac{F_0}{m}\right)\cos(\omega t)}$$

Our goal is to find a **steady state solution** (the solution that does not die out when $t \to \infty$). We consider seperate cases.

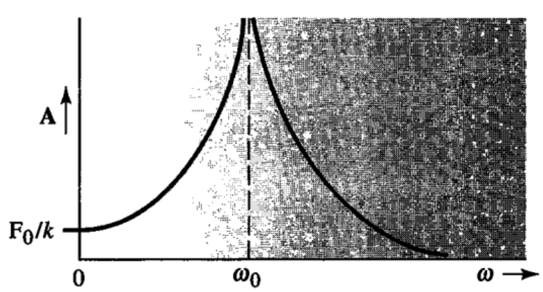
Case $\gamma = 0$: We guess that

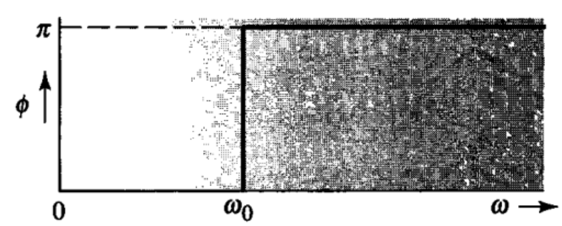
$$x(t) = A\cos(\omega t - \phi) \qquad (A > 0)$$

$$\ddot{x} + w_0^2 x = \left(\frac{F_0}{m}\right)\cos(\omega t) \quad \Longrightarrow \quad (-A\omega^2 + A\omega_0^2)\cos(\omega t - \phi) = \left(\frac{F_0}{m}\right)\cos(\omega t)$$

$$\Longrightarrow \quad A\cos(\omega t - \phi) = \frac{F_0/m}{\omega_0^2 - \omega^2}\cdot\cos(\omega t)$$

$$\Longrightarrow \quad \begin{cases} \omega < \omega_0 \quad \Longrightarrow \quad A = \frac{F_0/m}{\omega_0^2 - \omega^2} \\ \\ \omega > \omega_0 \quad \Longrightarrow \quad A = \frac{F_0/m}{\omega^2 - \omega_0^2} \end{cases}$$





Case $\gamma > 0$:

We assume that $F = F_0 e^{i\omega t}$:

$$\ddot{x} + 2\gamma\dot{x} + w_0^2 x = \left(\frac{F_0}{m}e^{iwt}\right)$$

Try:

$$x(t) = Ae^{i(wt-\phi)}$$

$$\implies \quad [-w^2 + 2\gamma wi + w_0^2]Ae^{i(wt-\phi)} \quad = \quad \frac{F_0}{m}e^{iwt}$$

$$A((w_0^2 - w^2) + 2\gamma wi) \quad = \quad \frac{F_0}{m}e^{i\phi} \quad = \quad \frac{F_0}{m}(\cos(\phi) + i\sin(\phi))$$

We take the real and imaginary roots:

$$\begin{cases} A(w_0^2 - w^2) = \frac{F_0}{m}\cos\phi \\ A \cdot 2\gamma w = \frac{F_0}{m}\sin\phi \end{cases}$$

We divide to get:

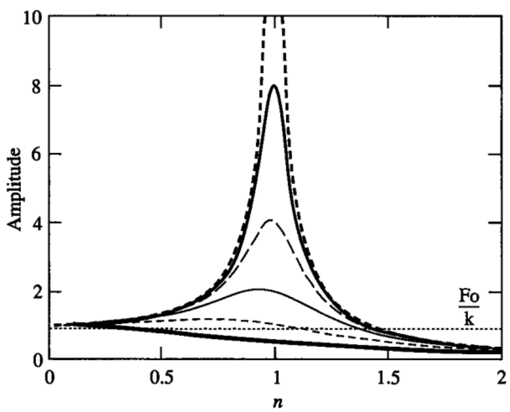
$$\boxed{\tan\phi = \frac{2\gamma w}{w_0^2 - w^2}}$$

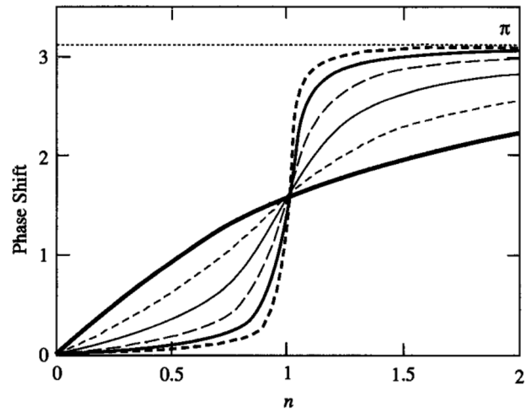
(Define the domain of $\phi \in [0, \pi]$). We now add the squares:

$$\left(\frac{F_0}{m}\right)^2 = A^2[(w_0^2 - w^2)^2 + 4\gamma^2 w^2]$$

$$\implies \quad \boxed{A(w) = \frac{F_0/m}{[(w_0^2 - w^2)^2 + 4\gamma^2 w^2]^{1/2}}}$$

The graphs below plot amplitude $A/(F_0/k)$ and phase shift ϕ for different values of γ as a function of $n = \omega/\omega_0$





$\implies$ our previous result when $\gamma \to 0$.

Resonance frequency? $$\left.\frac{dA(w)}{dw}\right|_{w=w_r} = 0$$

$$\implies \frac{-1/2}{[(w_0^2 - w^2)^2 + 4\gamma^2 w^2]^{3/2}} \cdot \frac{d}{dw}[(w_0^2 - w^2)^2 + 4\gamma^2 w^2] = 0$$

$$2 \times (-2w_r) \times (w_0^2 - w_r^2) + 8\gamma^2 w_r = 0$$

$$8\gamma^2 w_r = 4w_r(w_0^2 - w_r^2) \implies 2\gamma^2 = w_0^2 - w_r^2 \implies \boxed{w_r^2 = w_0^2 - 2\gamma^2}$$

Note: $w_r < w_0$ and $w_r^2 = w_d^2 - \gamma^2$

Also, we have no resonance for $\gamma > w_0/\sqrt{2}$ (w_r is complex in this case!) because $A(w)$ decreases monotonically with w:

$$A(w)|_{\gamma=w_0/2} = \frac{F_0/m}{[(w_0^2 - w^2)^2 + \frac{4w_0^2}{2}w^2]^{1/2}} = \frac{F_0/m}{[w_0^4 + w^4 - 2w_0^2w^2 + 2w_0^2w^2]^{1/2}}$$
$$= \frac{F_0/m}{[w_0^4 + w^4]^{1/2}}$$

and hence we have no resonance!

Maximum Amplitude, Quality Factor

$$A_{max} = A(w = w_r) = \frac{F_0/m}{[(w_0^2 - w_0^2 + 2\gamma^2)^2 + 4\gamma^2(w_0^2 - 2\gamma^2)]^{1/2}}$$

$$= \frac{F_0/m}{[4\gamma^4 - 8\gamma^4 + 4\gamma^2 w_0^2]^{1/2}} = \frac{F_0/m}{2\gamma[w_0^2 - \gamma^2]^{1/2}}$$

If we suppose that $\gamma << w_0$ we get that

$$\boxed{A_{max} \approx \frac{F_0}{2\gamma m w_0}} \propto \frac{1}{\gamma} \to \infty \qquad \text{when } \gamma \to 0$$

For $\gamma << w_0$, $\quad w_0^2 - w^2 = (w_0 + w)(w_0 - w) \approx 2w_0(w_0 - w)$

$$4\gamma^2 w^2 \approx 4\gamma^2 w_0^2$$

$$\implies \quad A(w) \approx \frac{F_0/m}{\sqrt{4w_0^2(w_0^2 - w^2) + 4\gamma^2 w_0^2}} = \frac{F_0}{2mw_0} \cdot \frac{1}{\sqrt{(w_0^2 - w^2) + \gamma^2}}$$

$$\boxed{A(w) \approx \frac{A_{max}\gamma}{\sqrt{(w_0^2 - w^2) + \gamma^2}}}$$

Note: $A^2(w = w_0 \pm \gamma) = (A_{max}/\sqrt{2})^2 = {A_{max}}^2 \implies (\Delta w) = 2\gamma$ (width of resonance)

Recall that

$$P = \frac{w_d}{2\gamma} \approx \frac{w_0}{2\gamma} = \frac{w_0}{\Delta w}$$

$$\implies \quad \boxed{\frac{\Delta w}{w_0} \approx \frac{1}{Q}}$$

The "sharpness" of the resonance can be quantified by how high the quality factor is!

$$\text{Also,} \quad \frac{A_{max}}{A(w=0)} = \frac{\frac{F_0}{2mw_0\gamma}}{\frac{F_0}{mw_0^2}} = \frac{w_0}{2\gamma} = Q \quad \implies \quad \boxed{\frac{A_{max}}{A(w=0)} = Q}$$

<u>At resonance, transfer of energy is "optimal":</u>

$P = Fv$

$$x(t) = A(w)\cos(wt - \phi)$$

When $w = w_r$ and $\phi = \pi/2$ we get

$$x(t) = A(w_r)\cos(w_r t - \pi/2) \quad = \quad A(w_r)\sin(w_r t)$$
$$\dot{x}(t) = w_r A(w_r)\cos(w_r t)$$

$\dot{x}$ is in phase with $F = F_0\cos(wt) \implies Fv = F_0 w_r A(w_r)\cos^2(w_r t)$. Hence the optimal way to push a swing is to push it when its velocity is maximum!

$$\boxed{< Fv >_{get-optimum} \quad = \quad \frac{F_0 w_r A(w_r)}{2}}$$