Lecture 8: Non-Linear Oscillator, Chaos

Consider a non-linear restoring force

$$F(x) = -kx + \epsilon(x) = -kx + \epsilon_2(x) + \epsilon_3(x) + \epsilon_4(x) + ...$$

Let us assume that

$$\begin{cases} V(-x) = +V(x) & \text{(INVERSION SYMMETRY)} \\ F(-x) = -F(x) & \text{and keep only up to third order} \end{cases}$$

$$m\ddot{x} = -kx + \epsilon_3 x^3$$

$$\boxed{\ddot{x} + w_0^2 x = \lambda x^3} \quad \text{where} \quad \lambda = \epsilon_3/m$$

Method of successive approximations: We try

$$x^{(0)}(t) = A\cos(wt) \qquad \text{“zeroth order” trial}$$

$$\Longrightarrow \; (w_0^2 - w^2)A\cos(wt) \;=\; \lambda A^3 \cos^3(wt) \;=\; \lambda A^3 \left(\frac{3}{4}\cos(wt) + \frac{1}{4}\cos(3wt)\right)$$

$$\Longrightarrow \; \left(w_0^2 - \frac{3}{4}\lambda A^2 - w^2\right) A\cos(wt) \;=\; \frac{\lambda A^3}{4}\cos(3wt)$$

The zeroth order approximation for w is obtained by setting the brackets to zero:

$$\Longrightarrow \quad \boxed{w = \sqrt{w_0^2 - \frac{3}{4}\lambda A^2}} \quad \text{This should be valid for small A and small } \lambda$$

The actual frequency of a non linear oscillator <u>depends on amplitude</u>

The next level of approximation is to take

$$x^{(1)}(t) = A\cos(wt) + B\cos(3wt)$$

We now let $\theta = wt$. Hence we have

$$\lambda(x^{(1)})^3 = \lambda(A^3\cos^3(\theta) + 3A^2\cos^2(\theta)B\cos(3\theta) + 3A\cos(\theta)B^2\cos^2(3\theta) + B^3\cos^3(3\theta))$$

$$= \lambda\left[A^3\left(\frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta)\right) + 3A^2B\left(\frac{1}{4}\cos(\theta) + \frac{1}{2}\cos(3\theta) + \frac{1}{4}\cos(5\theta)\right)\right.$$
$$\left.+3AB^2\left(\frac{1}{2}\cos(\theta) + \frac{1}{4}\cos(5\theta) + \frac{1}{4}\cos(7\theta)\right) + B^3\left(\frac{3}{4}\cos(3\theta) + \frac{1}{4}\cos(9\theta)\right)\right]$$

$$\implies \lambda(x^{(1)})^3 = \frac{3}{4}\lambda A^3\cos(\theta) + \frac{1}{4}\lambda A^3\cos(3\theta) + O(\lambda B, \lambda B^2, \lambda B^3)$$

Later, we will show that $B \propto \lambda$ and therefore that the last term is $O(\lambda^2)$!

$$\boxed{\ddot{x} + w_0^2 x = \lambda x^3}$$

$$\implies \quad \left(w_0^2 - \frac{3}{4}\lambda A^2 - w^2\right)A\cos(wt) + \left(Bw_0^2 - 9Bw^2 - \frac{1}{4}\lambda A^3\right)\cos(3wt) \;=\; O(\lambda^2) \approx 0$$

We examine the expression above. The first term is the same as before:

$$\boxed{w^2 = w_0^2 - \frac{3}{4}\lambda A^2}$$

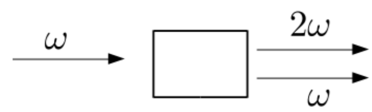
The second term implies that

$$B({w_0}^2 - 9w^2) = \frac{1}{4}\lambda A^3 \quad \implies \quad B = \frac{1}{4}\cdot\frac{\lambda A^3}{(w_0^2 - 9w_0^2 + (27/4)\lambda A^2)} \approx -\frac{\lambda A^3}{32w_0^2}$$

$$\implies \quad \boxed{x^{(1)}(t) = A\cos(wt) - \frac{\lambda A^3}{32w_0^2}\cos(3wt)} \quad \implies \quad \text{3rd Harmonic!}$$

What if $\epsilon_2 x^2$ is kept? In that case, we'd get a 2nd harmonic.

Experiment: 2nd harmonic generator, lasers on crystals:



Probes whether the crystal has an inversion center or not.

$$x^{(2)}(t) = A\cos(wt) + B\cos(3wt) + c\cos(5wt) \qquad c \propto \lambda^2$$

Non linear effects include that the period depends on amplitude, and that higher harmonics are present.

Example: Simple Pendulum:

$$\ddot{\theta} + \left(\frac{g}{l}\right)\sin(\theta) \;=\; 0 \quad\Longrightarrow\quad \ddot{\theta} + \frac{g}{l}\theta = \frac{1}{3!}\left(\frac{g}{l}\right)\theta^3$$

The result above is obtained by expanding the taylor series for $\sin(\theta)$. Rewriting g/l as w_0^2 yields

$$\ddot{\theta} + w_0^2\theta \;=\; \frac{w_0^2}{3!}\theta^3$$

We write $w_0^2/3! = \lambda$.

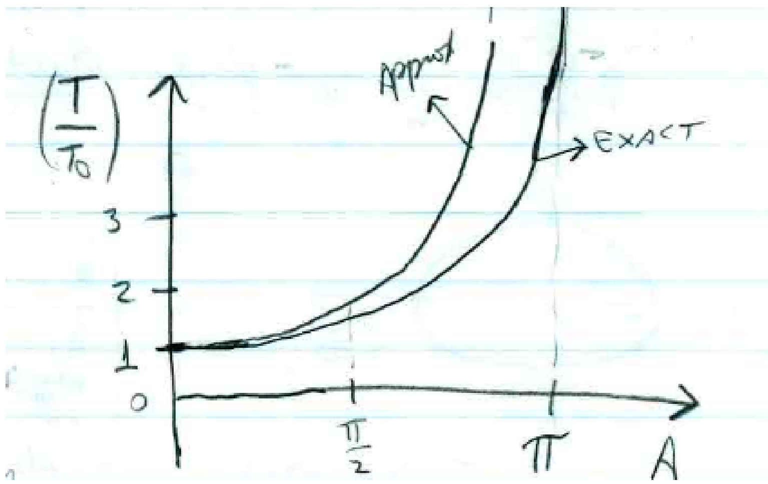
$$\Longrightarrow \quad w^2 = w_0^2 - \frac{3}{4}\lambda A^2$$

$$w^2 = w_0^2 - \frac{3}{4}\frac{w_0^2}{6}A^2 \;=\; w_0^2\left[1 - \frac{A^2}{8}\right]$$

$$T = \frac{2\pi}{w} = \frac{2\pi}{w_0}\frac{1}{\sqrt{1-(A^2/8)}} \quad\Longrightarrow\quad \boxed{\left(\frac{T}{T_0}\right)_{approx} \;=\; \frac{1}{\sqrt{1-(A^2/8)}}} > 1$$

How good is this approximation? Using energy conservation, we can show that

$$\left(\frac{T}{T_0}\right)_{exact} = \frac{\sqrt{2}}{\pi}\int_0^A \frac{d\theta}{\sqrt{\cos(\theta)-\cos(A)}}$$



The approximate solution goes to $t = \infty$ for $A = \sqrt{8} = 2.83 < \pi$. The exact solution goes to $t = \infty$ at $A = \pi$.

The approximation is very good for $A < \pi/2$. We will now examine the double pendulum and discuss chaos (sensitivity to initial conditions).

Chaos

Pendulum Subject to an Oscillatory Force

$$\ddot{\theta} + \gamma\dot{\theta} + {w_0}^2\sin(\theta) = \alpha\cos(w_f t)$$

We transform this equation into 3 first order ones:

$$\begin{cases} \dot{\theta} = y \\ \dot{y} = -\gamma y = {w_0}^2\sin(\theta) + \alpha\cos(z) \\ \dot{z} = w_f \implies z = w_f t \end{cases}$$

Numerical Solutions: $w_f = (2/3)w_0 \quad , \quad \gamma = (1/2)w_0$

(i) $\boxed{\alpha = 0.9{w_0}^2}$

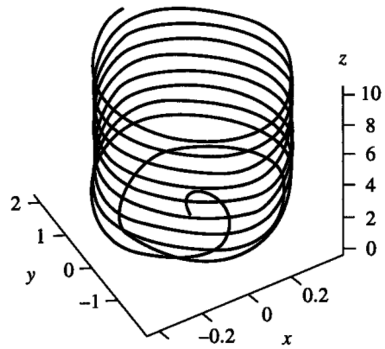
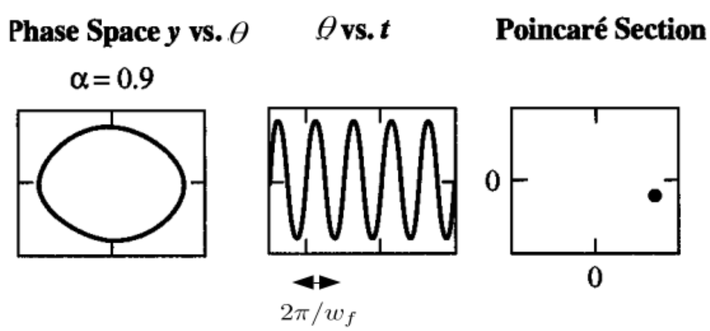


Figure 3.8.2 Three-dimensional phase-space plot of a driven, damped simple pendulum. The driving parameter is $\alpha = 0.9$. The driving angular frequency ω and damping parameter γ are $\frac{2}{3}$ and $\frac{1}{2}$ respectively. Coordinates plotted are $x = \theta/2\pi$, $y = \dot{\theta}$, $z = \omega t/2\pi$.

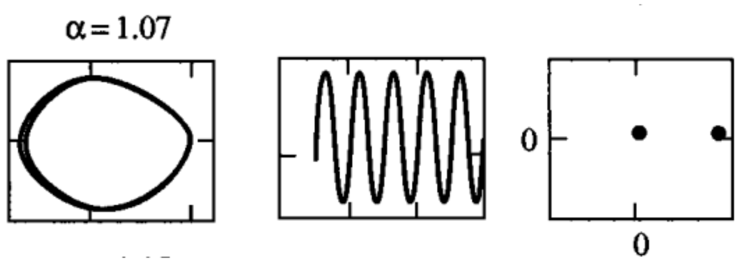
$\underline{\theta = 0 \quad , \quad \dot{\theta} = 0 :}$

After a transient, $t \sim 1/\gamma$. There is periodic behaviour at frequency w_f.



Stable, repeatable, periodic

(ii) $\boxed{\alpha = 1.07{w_0}^2}$ $\implies$ "Period Doubling"



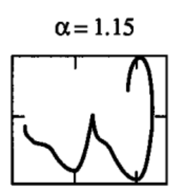
Period doubling! The system repeats itself every $t = 2 \times 2\pi/w$!

(iii) $\boxed{\alpha = 1.15{w_0}^2}$

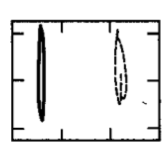
We define two starting coordinates:

$$\begin{cases} \theta_1 = -0.9\pi \quad , \quad \dot{\theta}_1 = 0.54660w_0 \\ \theta_2 = -0.9\pi \quad , \quad \dot{\theta}_2 = 0.54661w_0 \end{cases}$$

Note how similar the starting conditions are. Observe the phase space y vs. θ below

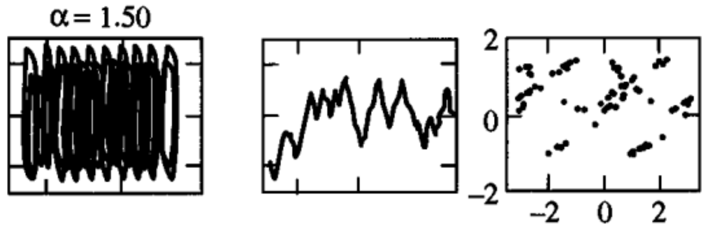


Trajectories 1 and 2 coincide for the first ~ 3 cycles. However, after 98 cycles, 1 and 2 are completely different. This extreme sensitivity to initial conditions leads to **unpredictability**.



$\rightarrow$ after 98 cycles, system does not have a single period

(iv) $\boxed{\alpha = 1.50{w_0}^2}$ $\Longrightarrow$ Never repeats itself!



For $\alpha > \alpha_{chaos}$, the system never repeats itself: