Lecture 9: Motion in 3d

Newton's laws imply that

$$\begin{cases} \vec{p} = m\vec{v} \\ \vec{F} = d\vec{p}/dt \end{cases}$$

$$\implies \quad \vec{F}\cdot\vec{v} \;=\; \frac{\vec{p}}{dt}\cdot\vec{v} \;=\; \frac{d}{dt}\left(m\vec{v}^2\right) \;=\; \frac{d}{dt}(T)$$

Where we define T as **kinetic energy**. We now have that

$$\int_i^f dt\vec{F}\cdot\frac{d\vec{r}}{dt} \;=\; \int_i^f \frac{d}{dt}(T)dt$$

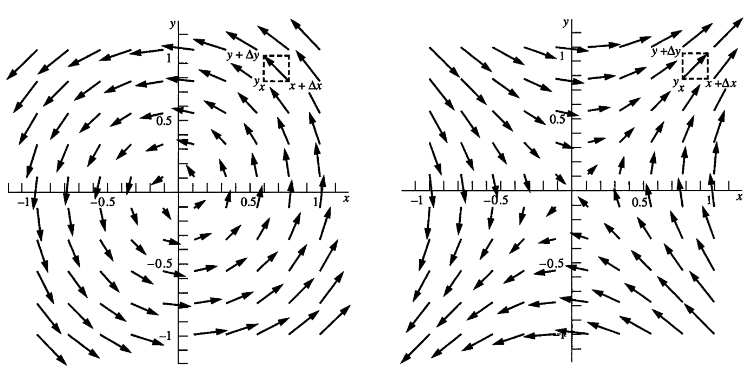
Note how the dt's cancel out on each side of the equation. It follows that

$$\int_{\vec{r_a}}^{\vec{r_b}} \vec{F}\cdot d\vec{r} \;=\; (T_f - T_i) \;=\; \Delta T \qquad \text{(Work-Kinetic Energy Theorem)}$$

What is the criteria on $\vec{F}$ such that the work done on the LHS can be written as potential energy? For this we consider different types of force fields.

Conservative vs Non-Conservative Force Fields

Consider $\vec{F}(\vec{r}) = b(-y,x)$ (left) and $\vec{F}(\vec{r}) = b(y,x)$ (right)



For which force fields can we define a potential function such that

$$V(\vec{r_f}) - V(\vec{r_i}) \;=\; -\int_{\vec{r_i}}^{\vec{r_f}} \vec{F}\cdot d\vec{r}$$

Note: This definition implies $-(V_f - V_i) = T_f - T_i \quad \implies \quad V_i + T_i = V_f + T_f$ which is the law of conservation of energy!

To be able to describe $\vec{F}$ by a potential $V(\vec{r})$, we must have:

$$\underset{\text{Path A}}{\int_{\vec{r_i}}^{\vec{r_f}} \vec{F}\cdot d\vec{r}} \;=\; \underset{\text{Path B}}{\int_{\vec{r_i}}^{\vec{r_f}} \vec{F}\cdot d\vec{r}}$$

Otherwise, the definition of V below does not depend just on $\vec{r_i}, \vec{r_f}$, but also on the path. In other words, V must be uniquely defined!

$$\underset{\text{Path A}}{\int_{\vec{r_i}}^{\vec{r_f}} \vec{F}\cdot d\vec{r}} \;-\; \underset{\text{Path B}}{\int_{\vec{r_i}}^{\vec{r_f}} \vec{F}\cdot d\vec{r}} \;=\; 0$$

$$\Longrightarrow \quad \oint \vec{F}\cdot d\vec{r} \;=\; 0$$

The integral with the circle on it simply means “for all closed paths.”

Stoke’s Theorem:

$$\oint \vec{F}\cdot d\vec{r} \;=\; \iint \vec{\nabla}\times\vec{F}\cdot\hat{m}dA \;=\; 0 \iff \vec{\nabla}\times\vec{F} = 0 \qquad \text{at all points}$$

Recall that the second integral is the area enclosed by the path and that $\vec{\nabla} =< \partial x, \partial y, \partial z >$. One way to get $\vec{\nabla}\times\vec{F} = 0$ at all points is to have

$$\vec{F} = -\vec{\nabla}V \quad , \quad \text{because} \quad \vec{\nabla}\times\vec{F} \;=\; -\vec{\nabla}\times\vec{\nabla}V \;=\; 0$$

But there exists some force fields $\vec{F}$ that can be written as $\vec{F} = -\vec{\nabla}V$ but that $\vec{\nabla}\times\vec{F}$ does not exist (because second derivatives d^2V/dx^2 does not exist.

$\Longrightarrow$ See Amer. J. Phys. 37,616 (1969).

Back to our Example

$$\vec{\nabla} \times \vec{F} \;=\; \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} \;=\; (\partial_y F_z - \partial_z F_y)\hat{i} + (\partial_z F_x - \partial_x F_z)\hat{j} + (\partial_x F_y - \partial_y F_x)\hat{k}$$

$$= \;\epsilon_{ijk}\partial_g \cdot F_k \qquad \text{(The Levi-Coveta Tensor)}$$

Is $\vec{F}_1(\vec{n}) = b(-y, x)$ conservative?

$$\vec{\nabla} \times \vec{F}_1 = (\partial_x F_y - \partial_y F_x)\hat{k} \;=\; (b - (-b))\hat{k} \;=\; 2b\hat{k} \quad \neq \quad \vec{0} \qquad \text{Non Conservative}$$

Is $\vec{F}_2(\vec{n}) = b(y, x)$ conservative?

$$\vec{\nabla} \times \vec{F}_2 = (\partial_x F_y - \partial_y F_x)\hat{k} \;=\; (b - (b))\hat{k} \;=\; \vec{0} \qquad \text{Conservative}$$