

## Lecture 10: Motion of a Projectile and Motion of a Particle in Electromagnetic Fields

Forces of the separable type:

$$\vec{F} = F_x(x)\hat{i} + F_y(y)\hat{j} + F_z(z)\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ F_x & F_y & F_z \end{vmatrix} = 0$$

### Projectile motion with linear air resistance

$$m \frac{d^2\vec{r}}{dt^2} = -m\gamma\vec{v} - mg\hat{k}$$

$$\Rightarrow \begin{cases} \ddot{x} = -\gamma\dot{x} \\ \ddot{y} = -\gamma\dot{y} \\ \ddot{z} = -\gamma\dot{z} - g \end{cases}$$

Assume  $x_0 = y_0 = z_0 = 0$ ,  $\dot{x}_0 \neq 0$ ,  $\dot{z}_0 \neq 0$ ,  $\dot{y}_0 = 0$ . Solving for  $\dot{x}$  and  $\dot{y}$  is straightforward;  $\dot{z}$  is harder:

**To find  $\dot{z}$ :**

Define  $w = \dot{z}$  and solve the 1st order equation

$$\dot{w} + \gamma w = -g$$

Multiply both sides by integrating factor  $X(t)$ :

$$X\dot{w} + \gamma Xw = -gX, \text{ demand that}$$

$$\frac{d}{dt}(Xw) = -gX$$

$$X\dot{w} + \dot{X}w = -gX \implies \dot{X} = \gamma X \implies X(t) = e^{\gamma t}$$

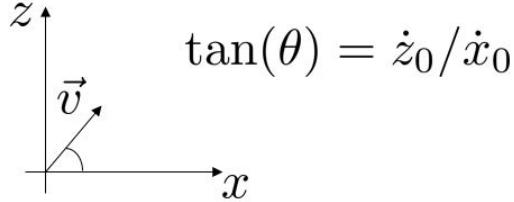
$$\begin{aligned} \implies \int \frac{d}{dt}(Xw)dt &= -g \int_0^t dt X(t) \\ X(t)w(t) - X(0)w(0) &= -\frac{g}{\gamma}(e^{\gamma t} - 1) \end{aligned}$$

Now multiplying both sides by  $e^{-\gamma t}$  yields

$$w(t) - e^{-\gamma t} \dot{z}_0 = -\frac{g}{\gamma}(1 - e^{-\gamma t}) \implies \boxed{\dot{z} = \dot{z}_0 e^{-\gamma t} - \frac{g}{\gamma}(1 - e^{-\gamma t})}$$

We now have that

$$\begin{cases} \dot{x} = \dot{x}_0 e^{-\gamma t} \\ \dot{y} = \dot{y}_0 e^{-\gamma t} \\ \dot{z} = \dot{z}_0 e^{-\gamma t} - (g/\gamma)(1 - e^{-\gamma t}) \end{cases}$$



We choose  $\dot{y} = 0$  so  $y \equiv 0$ . Integrating again yields

$$\begin{aligned} \begin{cases} x = (\dot{x}_0/\gamma)(1 - e^{-\gamma t}) \\ y = 0 \\ z = \left(\frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2}\right)(1 - e^{-\gamma t}) - \frac{gt}{\gamma} \end{cases} \\ \implies \boxed{\vec{r}(t) = \left(\frac{\vec{v}_0}{\gamma} + \frac{g}{\gamma^2} \hat{k}\right)(1 - e^{-\gamma t}) - \frac{gt}{\gamma} \hat{k}} \end{aligned}$$

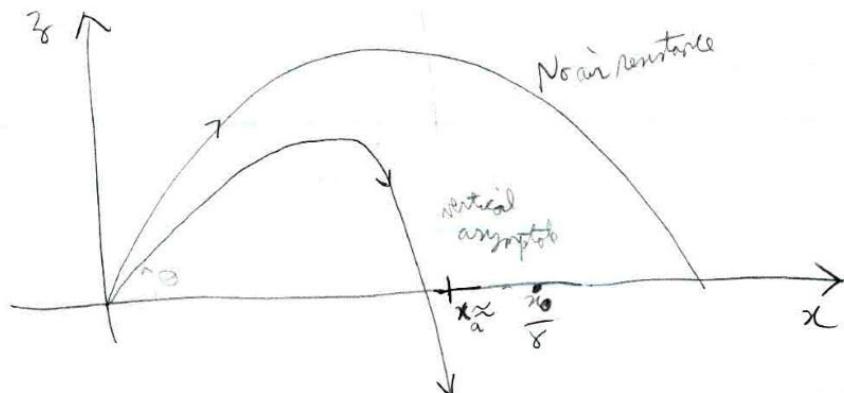
**Check limit**  $\gamma \rightarrow 0$

$$\begin{aligned}\vec{r}(t) &\rightarrow \left( \frac{\vec{v}_0}{\gamma} + \frac{g}{\gamma^2} \hat{k} \right) \left( \gamma t - \frac{\gamma^2 t^2}{2} + \sigma(\gamma^3) \right) - \frac{gt}{\gamma} \hat{k} \\ &= \vec{v}_0 t - \vec{v}_0 \frac{\gamma t^2}{2} + \sigma(\gamma^2) + \frac{gt}{\gamma} \hat{k} - \frac{gt^2}{2} \hat{k} + \sigma(\gamma) - \frac{gt}{\gamma} \hat{k} \\ \vec{r}(t)_{\gamma \rightarrow 0} &= \vec{v}_0 t - \frac{gt^2}{2} \hat{k} + \sigma(\gamma) \quad \text{Checks!}\end{aligned}$$

Interestingly, as  $t \rightarrow \infty$  and hence  $t \gg 1/\gamma$  we have that

$$\vec{r}(t) \approx \frac{\vec{v}_0}{\gamma} - \frac{gt}{\gamma} \hat{k}$$

The first term  $\vec{v}_0/\gamma$  implies that the air resistance has stopped motion along x! The second term  $-gt/\gamma$  represents the terminal velocity  $\times$  time.



## Horizontal Range?

Eliminate  $t$  and set  $z = 0$  :  $x = (\dot{x}_0/\gamma)(1 - e^{-\gamma t})$

$$e^{-\gamma t} = 1 - \frac{\gamma x}{\dot{x}_0} \implies t = -\frac{1}{\gamma} \left( 1 - \frac{\gamma x_{max}}{\dot{x}_0} \right) , \text{ and}$$

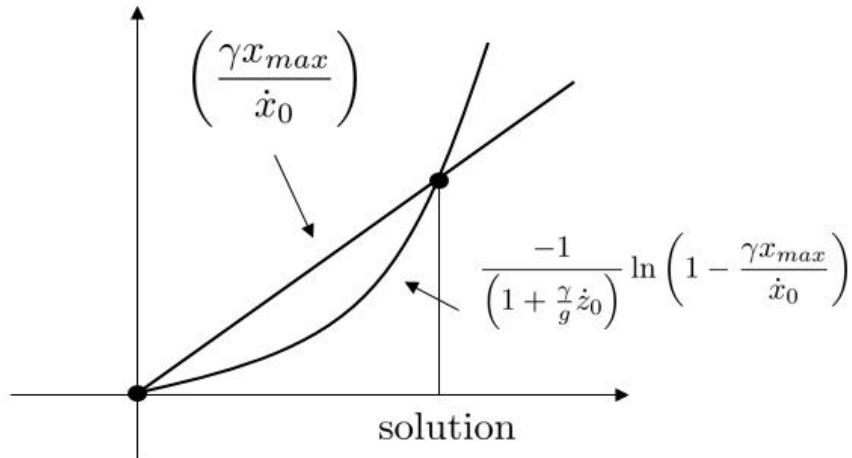
$$z = 0 = \left[ \left( \frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2} \right) \frac{\gamma x_{max}}{\dot{x}_0} + \frac{g}{\gamma^2} \ln \left( 1 - \frac{\gamma x_{max}}{\dot{x}_0} \right) \right] = 0$$

Note that  $\gamma x_{max}/\dot{x}_0$  is always  $< 1$ .

We have a transcendental expression; we solve numerically:

$$\left( \frac{\gamma x_{max}}{\dot{x}_0} \right) = \frac{-g/\gamma^2}{\left( \frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2} \right)} \ln \left( 1 - \frac{\gamma x_{max}}{\dot{x}_0} \right)$$

$$\left( \frac{\gamma x_{max}}{\dot{x}_0} \right) = \frac{-1}{\left( 1 + \frac{\gamma}{g} \dot{z}_0 \right)} \ln \left( 1 - \frac{\gamma x_{max}}{\dot{x}_0} \right)$$



**Analytic Solution:** Valid when  $\gamma x_{max}/\dot{x}_0 \ll 1$ .

We define  $u = \gamma x_{max}/\dot{x}_0$ . Hence

$$u = \frac{-1}{\left(1 + \frac{\gamma}{g}\dot{z}_0\right)} \ln(1 - u)$$

After Taylor Expansion we have  $\ln(1 - u) = -u - u^2/2 - u^3/3 - \dots$

It follows that

$$\begin{aligned} \left(1 + \frac{\gamma}{g}\dot{z}_0\right)u &= \left[u + \frac{u^2}{2} + \frac{u^3}{3} + O(u^4)\right] \\ \frac{\gamma}{g}\dot{z}_0 &= \frac{u}{2} + \frac{u^2}{3} + O(u^3) \\ \boxed{u^2 + \frac{3}{2}u - \frac{3\gamma\dot{z}_0}{g} = 0} \end{aligned}$$

We now have that

$$u_{\pm} = \frac{1}{2} \left( -\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 4 \left(\frac{-3\gamma\dot{z}_0}{g}\right)} \right)$$

Using the + sign gives us positive u

$$u = \frac{3}{4} \left( -1 + \sqrt{1 + \frac{16}{3} \left(\frac{\gamma\dot{z}_0}{g}\right)} \right)$$

and since  $\sqrt{1+x} \approx 1 + 1/2x - 1/8x^2 + \dots$  we have

$$u \approx \frac{3}{4} \left( -1 + \left( 1 + \frac{8}{9} \frac{\gamma\dot{z}_0}{g} - \frac{1}{8} \left(\frac{16}{3} \frac{\gamma\dot{z}_0}{g}\right)^2 \right) \right) \approx \frac{2\gamma\dot{z}_0}{g} - \frac{8}{3} \left(\frac{\gamma\dot{z}_0}{g}\right)^2$$

$$x_{max} = \frac{\dot{x}_0}{\gamma} u \implies x_{max} \approx \frac{2\dot{x}_0 \dot{z}_0}{g} - \frac{8}{3} \frac{\gamma}{g^2} \dot{x}_0 \dot{z}_0^2 + \dots$$

We introduce  $\tan(\theta) = (\dot{z}_0/\dot{x}_0)$  and since  $v_0^2 = \dot{z}_0^2 + \dot{x}_0^2$  we have

$$\sin(\theta) = \frac{\dot{z}_0}{v_0} \quad \cos(\theta) = \frac{\dot{x}_0}{v_0}$$

It follows that

$$x_{max} \approx \frac{v_0^2}{g} \sin(2\theta) - \frac{8v_0^3}{3g^2} \cos(\theta) \sin^2(\theta) \gamma + \dots$$

This is valid when

$$\begin{aligned} \frac{\gamma x_{max}}{\dot{x}_0} &<< 1 \implies \frac{\gamma}{\dot{x}_0} \frac{2\dot{x}_0 \dot{z}_0}{g} << 1 \implies \frac{2\gamma v_0}{g} \sin(\theta) << 1 \\ &\implies \text{want } \boxed{\frac{2\gamma v_0}{g} << 1} \quad \text{for all angles} \end{aligned}$$

Hence we want either low  $\gamma$  or low  $v_0$ . To find the optimum angle:

$$\frac{dx_{max}}{d\theta} = 0 = \frac{2v_0^2}{g} \cos(2\theta) - \frac{8\gamma v_0^3}{3g^2} (-\sin^2(\theta) + 2\sin(\theta)\cos^2(\theta))$$

Note that  $-\sin^2(\theta) + 2\sin(\theta)\cos^2(\theta) = \sin(\theta)[\cos^2(\theta) + \cos(2\theta)]$

$$\begin{aligned} \implies \frac{2v_0^2}{g} \cos(2\theta) &= \frac{8\gamma v_0^3}{3g^2} \sin(\theta)(\cos^2(\theta) + \cos(2\theta)) \\ \implies \frac{\cos(2\theta)}{\sin(\theta)[\cos^2(\theta) + \cos(2\theta)]} &= \left( \frac{4\gamma v_0}{3g} \right) \end{aligned}$$

