# Lecture 11: Motion of Charged Particles in Electric and Magnetic Fields

**Electric Field**:

$$\vec{F} = q\vec{E} \quad \Longrightarrow \quad m\frac{d^2\vec{r}}{dt^2} = q\vec{E}$$

$$\vec{E} = E\hat{k} \quad \Longrightarrow \quad \begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = 0 \\ m\ddot{z} = qE \end{cases} \quad \Longrightarrow \vec{r}(t) = \vec{v}(t) + \frac{1}{2}\left(\frac{qE}{m}\right)\hat{k}t^2$$

**Magnetic Field**:

$$\vec{F} = q(\vec{v} \times \vec{B}) \qquad \text{(Lorentz force in S.I. units)}$$

$$\Longrightarrow \quad \text{acceleration always} \perp \vec{v} \quad \Longrightarrow \quad \text{speed is constant! (No tangential acceleration)}$$

Suppose that $\vec{B} = B\hat{k}$. It follows that

$$m\frac{d^2\vec{r}}{dt^2} = qB(\vec{v} \times \vec{k}) \;=\; qB\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & 1 \end{vmatrix} \;=\; qB(\dot{y}\hat{i} - \dot{x}\hat{j})$$

$$\Longrightarrow \quad \begin{cases} m\ddot{x} = qB\dot{y} \\ m\ddot{y} = -qB\dot{x} \\ m\ddot{z} = 0 \Longrightarrow \dot{z} = \dot{z}_0 \quad , \quad z = z_0 + \dot{z}_0 t \end{cases}$$

This is not separable but it is easy to solve:

$$m\int \frac{d\dot{x}}{dt}dt \;=\; qB\int \frac{dy}{dt}dt \quad\Longrightarrow\quad m(\dot{x}-\dot{x}_0)=qB(y-y_0) \quad\Longrightarrow\quad \boxed{\dot{x}=\frac{qB}{m}y+C_1}$$

$$m\int \frac{d\dot{y}}{dt}dt \;=\; -qB\int \frac{dx}{dt}dt \quad\Longrightarrow\quad m(\dot{y}-\dot{y}_0)=qB(x-x_0) \quad\Longrightarrow\quad \boxed{\dot{y}=-\frac{qB}{m}x+C_2}$$

We define the "cyclotron" frequency $w_c = qB/m$.

($q = e$ and $B = 1T \implies w_c = 10^{15} rad/s$. For a red laser, $w = 3 \times 10^{15} rad/s$ and $\lambda = 600nm$).

We take $d/dt$ of both sides:

$$\ddot{x} = w_c\dot{y} = w_c(-w_c x + C_2) \quad\Longrightarrow\quad \ddot{x} + w_c^2 x = w_c C_2$$

$$\ddot{y} = -w_c\dot{x} = -w_c(w_c y + C_1) \quad\Longrightarrow\quad \ddot{y} + w_c^2 y = -w_c C_1$$

We now define $a \equiv C_2/w_c$ and $b = C_1/w_c$.

$$\ddot{x} + w_c^2 x = w_c^2 a \quad\Longrightarrow\quad \boxed{x(t) = a + A\cos(w_c t + \theta_0)}$$

$$y = \frac{\dot{x} - C_1}{w_c} \;=\; \frac{\dot{x} + w_c b}{w_c} \;=\; \frac{-w_c A\sin(w_c t + \theta_0) + bw_c}{w_c} \quad\Longrightarrow\quad \boxed{y(t) = b - A\sin(w_c t + \theta_0)}$$

It follows that

$$\boxed{(x-a)^2 + (y-b)^2 = A^2}$$

This is a circle centered at (a,b) with radius A:

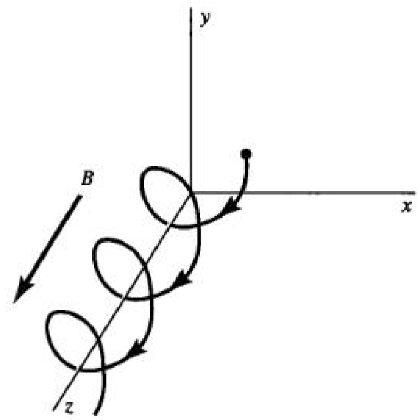
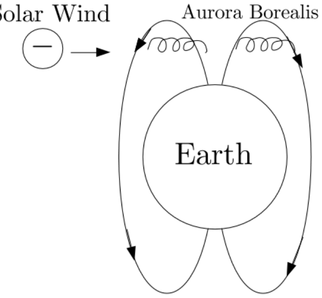



**Figure 4.5.1** The helical path of a particle moving in a magnetic field.

The path is a helix along $\vec{B}$ (if $\dot{z}_0$, the path is a closed circle in the xy plane!). In this way, the magnetic field "confines" the motion of charged particles:

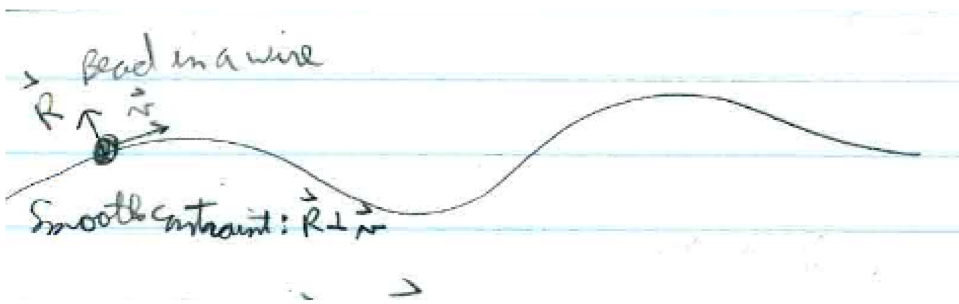



$$\dot{y} = -w_c A \cos(w_c t + \theta_0)$$

$$(\dot{x}^2 + \dot{y}^2) = w_c^2 A^2 = v_\perp^2 = v_0^2 \implies A = \frac{v_0}{qB/m} = \boxed{v_0 \frac{m}{qB}} \propto \frac{1}{B}$$

Note that $v_\perp^2 = v_0^2$ because $\vec{F}$ is perpendicular to $\vec{v}$ always. The result above implies that the radius of the circle is proportional to $1/B$.

## Constrained Motion of a Particle




$$m\frac{d\vec{r}}{dt} = \vec{F} + \vec{R} \quad \Longrightarrow \quad m\frac{d\vec{v}}{dt}\cdot\vec{v} \;=\; \vec{F}\cdot\vec{v} + \vec{R}\cdot\vec{v}$$

where $\vec{R}$ is the constraint. We suppose for now that $\vec{F}$ is conservative. Note that $\vec{R}\cdot\vec{v} = 0$ for a "smooth constraint" (no friction).

$$\Longrightarrow \quad \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2\right) \;=\; \vec{F}\cdot\vec{v} \quad \Longrightarrow \quad d\left(\frac{1}{2}m\vec{v}^2\right) = \vec{F}\cdot d\vec{r} = dW = -dV$$

$$\Longrightarrow \quad \frac{1}{2}mv^2 + V(\vec{r}) = E = \text{const}$$

**Example:** Particle Sliding on a smooth sphere. When does it detach?

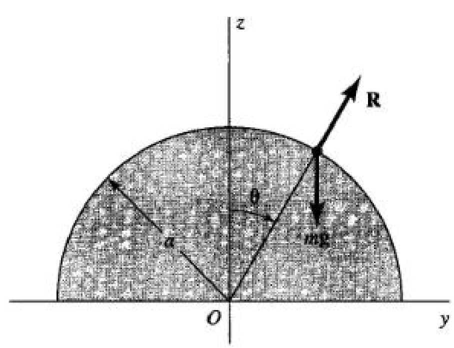



**Figure 4.6.1** A particle sliding on a smooth sphere.

$$\frac{1}{2}mv^2 + mgz = E = mga$$

The last equality is true because the particle starts from rest.

$$\Longrightarrow \quad \boxed{v^2 = 2g(a - z)}$$

In addition, we have

$$m\frac{d^2\vec{r}}{dt^2} = m\vec{g} + \vec{R}$$

<u>Radial:</u>

$$ma_r = -mg\cos\theta + R$$

$$m\left(-\frac{v^2}{a}\right) = -mg\left(\frac{z}{a}\right) + R \quad \Longrightarrow \quad R = \left(\frac{mg}{a}\right)z - \left(\frac{m}{a}\right)v^2$$

Since $v^2 = 2g(a - z)$ we have that

$$R(z) = mg\left(\frac{z}{a}\right) - \left(\frac{m}{a}\right)2g(a - z) = -2mg + 3mg\left(\frac{z}{a}\right)$$

$$\boxed{R(z) = mg\left(\frac{3z}{a} - 2\right)}$$

When does it detach? When $mg\cos\theta < mv^2/a \implies R < 0$. This occurs because the weight is not strong enough to hold the particle in a circular path.

Note that $R = 0$ for $\boxed{z = (2/3)a}$.