Lecture 12: Non-Inertial Reference Systems

σ is inertial reference system, σ' is not:

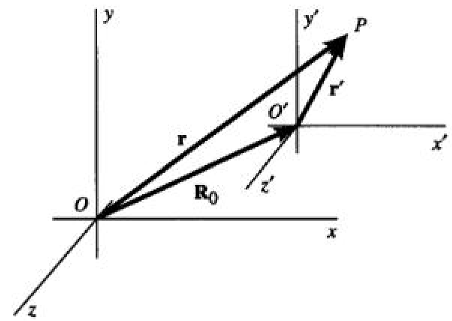


Figure 5.1.1 Relationship between the position vectors for two coordinate systems undergoing pure translation relative to each other.

$$\vec{r} = \vec{R}_\sigma + \vec{r}\,'$$

$$\frac{d}{dt}\vec{r} = \frac{d\vec{R}_\sigma}{dt} + \frac{d\vec{r}\,'}{dt} \quad \Longrightarrow \quad \vec{v} = \vec{V}_\sigma + \vec{v}\,'$$

$$\Longrightarrow \quad \vec{a} = \vec{a}_\sigma + \vec{a}\,'$$

Newton's Second law for σ:

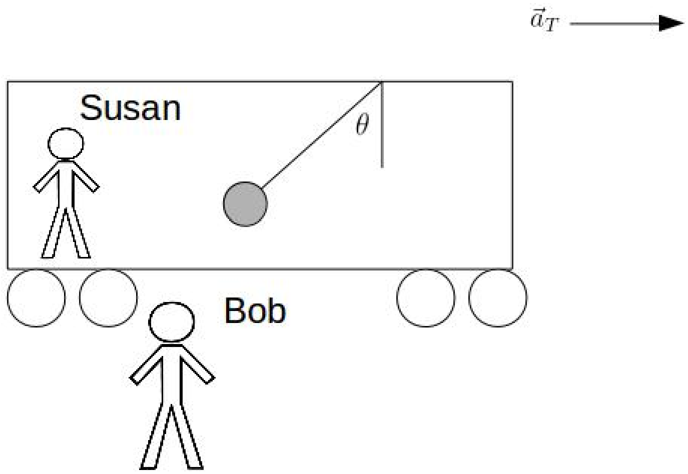
$$\begin{aligned} \Sigma\vec{F} &= m\vec{a} \\ &= m(\vec{a}_\sigma + \vec{a}\,') \end{aligned}$$

Newton's Second law in a non inertial system:

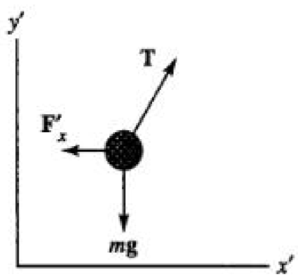
$$\Sigma\vec{F} - m\vec{a}_\sigma = m\vec{a}$$

We call $-m\vec{a}_\sigma$ the inertial force or "fictitious force" (yet it is quite real actually).

Example 1:



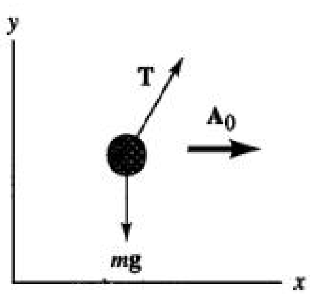
Susan Sees:



$$F_x' = -m\vec{a}_T$$

$$\begin{cases} T\cos\theta - mg = ma_y = 0 \\ T\sin\theta - ma_T = ma_x = 0 \end{cases} \implies \boxed{a_T = g\tan\theta}$$

Bob Sees:



$$\begin{cases} T\cos\theta - mg = ma_y = 0 \\ T\sin\theta = ma_x = ma_T \neq 0 \end{cases}$$

$$\implies \frac{mg}{\cos\theta}\sin\theta = ma_T$$

$$\boxed{g\tan\theta = a_T}$$

Example 2:
Consider a spaceship accelerating upwards with $\vec{a}_\sigma = \vec{g} = g\vec{j}$

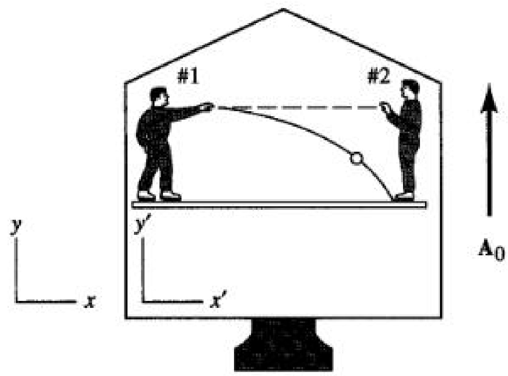


Figure 5.1.3 Two astronauts throwing a ball in a spaceship accelerating at $|\mathbf{A}_0| = |\mathbf{g}|$.

Two astronauts are 10m away from each other. Astronaut 1 throws a ball at astronaut 2 with constant velocity $\vec{v}_0 = v_0\hat{i}$. What is the minimum speed v_0 such that the ball reaches astronaut 2 without touching the floor?

(a): Non-Inertial Reference Frame

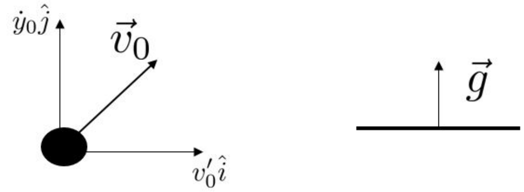
Force $-mg\hat{j}$ pulling everything downwards!

$$\begin{cases} x'(t) = \dot{x}_0 t \\ y'(t) = y'_0 - (1/2)gt^2 \end{cases} \implies y' = y'_0 - \frac{g}{2(\dot{x_0}')^2} \cdot x'^2$$

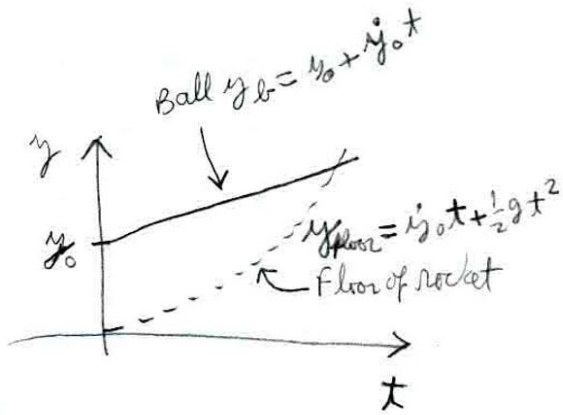
$$0 = y'_0 - \frac{g}{2(\dot{x_0}')^2} \cdot x'^2 \implies \frac{g}{2(\dot{x_0}')^2} \cdot x'^2 = y'_0$$

$$\implies \dot{x_0}' = \sqrt{\frac{g}{2y'_0}} \cdot x' = \sqrt{\frac{9.8}{2 \times 2}} \times 10 = \boxed{15.6m/s}$$

(b): Inertial Reference Frame



The ball is thrown diagonally with constant velocity and no acceleration. The floor of the rocket is accelerating upwards at $\vec{g}$.



We have that

$$\begin{cases} y_b = y_0 + \dot{y}_0 t \\ x_b = \dot{x}_0 t \end{cases} \qquad \begin{cases} y_{floor} = \dot{y}_0 t + (1/2) g t^2 \end{cases}$$

and hence it follows that

$$y_b = y_{floor} \implies y_0 + \dot{y}_0 t = \dot{y}_0 t + \frac{1}{2} g t^2 \implies t = \sqrt{\frac{2y_0}{g}}$$

$$x_b = \dot{x}_0 \sqrt{\frac{2y_0}{g}} \implies \boxed{\dot{x}_0 = \sqrt{\frac{g}{2y_0}} \cdot x_b}$$

Rotating Coordinate Systems

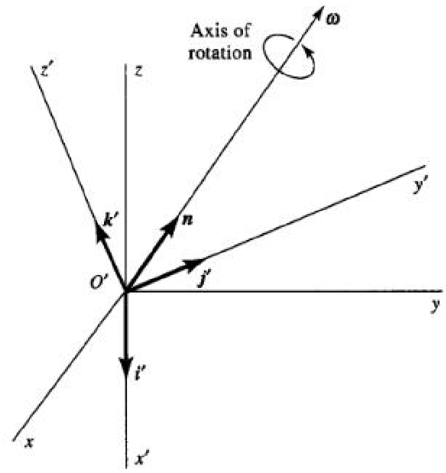


Figure 5.2.1 The angular velocity vector of a rotating coordinate system.

$\vec{w} = w\vec{n}$ is the angular velocity. The direction is given by the right hand rule. In addition

$$w = 2\pi/T$$

Transformation rules for the velocities? The origin is the same, therefore

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \;\; = \;\; x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$$

$$\frac{d}{dt}\vec{r} \;\; = \;\; v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \;\; = \;\; \left(\frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}'\right) + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

$$\Longrightarrow \quad \boxed{\vec{v} = \vec{v}\,' + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}}$$

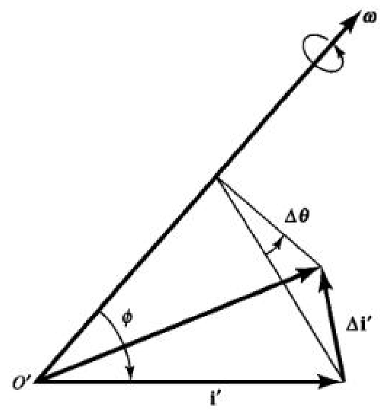


Figure 5.2.3 Change in the unit vector **i′** produced by a small rotation $\Delta\theta$.

$$\left|\frac{d\hat{i}'}{dt}\right| = \lim_{\Delta t\to 0}\frac{|\Delta\hat{i}'|}{\Delta t} = \sin\phi\frac{\Delta\theta}{\Delta t} = w\sin\phi$$

also, $\dfrac{d\hat{i}'}{dt}\perp\hat{i}',\vec{w}_0 \implies \boxed{\dfrac{d\hat{i}'}{dt}=\vec{w}\times\vec{i}\,'}$ $\left(\text{Similarily, } \dfrac{d\hat{j}'}{dt}=\vec{w}\times\vec{j}\,' \quad , \quad \dfrac{d\hat{k}'}{dt}=\vec{w}\times\vec{k}\,'\right)$

Therefore:

$$\boxed{\vec{v}=\vec{v}\,'+\vec{w}\times\vec{r}\,'}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed}=\left(\frac{d\vec{r}\,'}{dt}\right)_{not}+\vec{w}\times\vec{r}\,' = \left[\left(\frac{d}{dt}\right)_{not}+\vec{w}\times\right]\vec{r}\,'$$

Applies to any vector! For example, velocity:

$$\left(\frac{d\vec{v}}{dt}\right)_{fixed} = \left(\frac{d\vec{v}}{dt}\right)_{not}+\vec{w}\times\vec{v}$$

But $\vec{v}=\vec{v}\,'+\vec{w}\times\vec{r}\,'$:

$$\left(\frac{d\vec{v}}{dt}\right)_{fixed} = \left(\frac{d}{dt}\right)_{not}(\vec{v}\,'+\vec{w}\times\vec{r}\,')+\vec{w}\times(\vec{v}\,'+\vec{w}\times\vec{r}\,')$$

$$\vec{a}_{fixed} = \vec{a}\,' + \left(\frac{d\vec{w}}{dt}\right)_{not} \times \vec{r}\,' + \vec{w} \times \vec{w}\,' + \vec{w} \times \vec{v}\,' + \vec{w} \times (\vec{w} \times \vec{r}\,')$$

Note that $(d\vec{w}/dt)_{not} = (d\vec{w}/dt)_{fixed}$ since $\vec{w} \times \vec{w} = 0$. It follows that

$$\vec{a}_{fixed} = \vec{a}\,' + \dot{\vec{w}} \times \vec{r}\,' + 2\vec{w} \times \vec{v}\,' + \vec{w} \times (\vec{w} \times \vec{r}\,') + \vec{A}_\sigma$$

- $\dot{\vec{w}} \times \vec{r}\,'$ is the "transverse acceleration"
- $2\vec{w} \times \vec{v}\,'$ is the "coriolis acceleration "
- $\vec{w} \times (\vec{w} \times \vec{r}\,')$ is the "centripetal acceleration"
- $\vec{A}_\sigma$ is the "translational acceleration" if σ us accelerating with respect to σ' (origins do not match)

Note that

$$\vec{w} \times (\vec{w} \times \vec{r}\,') = (\vec{w} \cdot \vec{r}\,')\vec{w} - (\vec{w} \cdot \vec{w})\vec{r}' \equiv w^2[\vec{r}\,' - (\vec{w} \cdot \vec{r}\,')\hat{w}]$$

We call this the "curve of rotation."

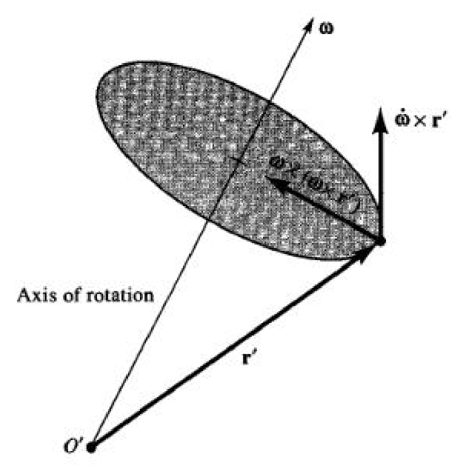


Figure 5.2.5 Illustrating the centripetal acceleration.