Lecture 13: Dynamics of a Particle in a Rotating Coordinate System

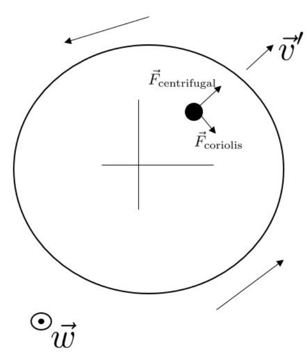
$$\vec{F} = m\vec{a} \quad \Longrightarrow \quad \vec{F}\,' = m\vec{a}\,'$$

$$\vec{F}\,' \;=\; \vec{F}_{physical} \;-\; m\vec{A}_0 \;-\; 2m\vec{w}\times\vec{v}\,' \;-\; m\vec{w}\times(\vec{w}\times\vec{r}\,') \;-\; m\dot{\vec{w}}\times\vec{r}\,' \;=\; m\vec{a}\,'$$

- $\vec{A}_0$ is the acceleration of the origin σ' with respect to σ.
- $-2m\vec{w}\times\vec{v}\,'$ is the **coriolis force**
- $-m\vec{w}\times(\vec{w}\times\vec{r}\,')$ is the **centrifugal force**
- $-m\dot{\vec{w}}\times\vec{r}\,'$ is the **transverse force**

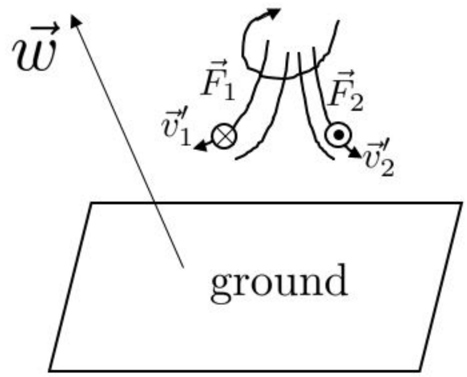
The coliolis force, centrifugal force, and traverse force make up the **internal forces**.

Inertial forces for a point moving outward in a merry go round:

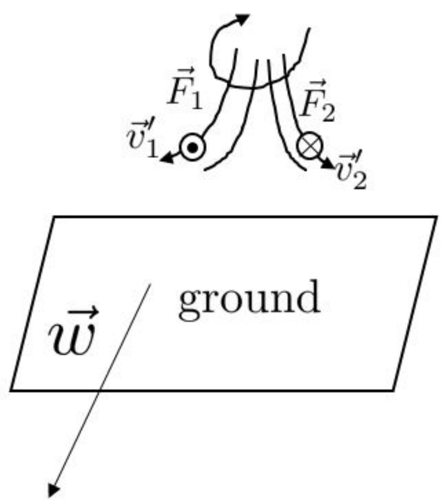


Rotation of Hurricanes

North Hemisphere: Clockwise



South Hemisphere: Anticlockwise



Example 1: A rolling wheel

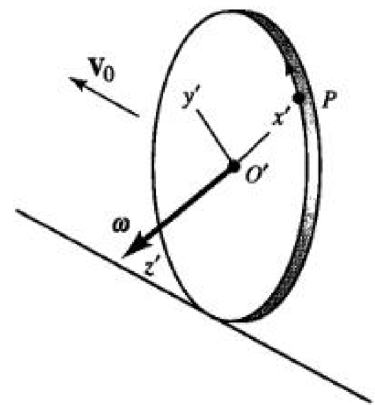


Figure 5.2.6 Rotating coordinates fixed to a rolling wheel.

Acceleration of point P with respect to the ground? Assume the wheel has a radius b.

$$\vec{w} = w\hat{k}\,' = \frac{v_0}{b}\hat{k}\,' \quad , \quad \begin{cases} \vec{r}\,' = b\hat{i}\,' \\ \vec{a}\,' = 0 \quad , \quad \vec{v}\,' = 0 \end{cases}$$

All terms vanish except centripetal!

<u>Transverse Rule</u>:

$$\vec{a} = \vec{w} \times (\vec{w} \times \vec{r}\,') = w^2 b\hat{k}\,' \times (\hat{k}\,' \times \hat{i}\,') = -w^2 b\hat{i}\,' \; = \; \boxed{\frac{v_0^2}{b}(-\hat{i}\,')}$$

Example 2: Bicycle with constant speed of radius ρ. Acceleration of the highest point on one of the wheels with respect to the ground?

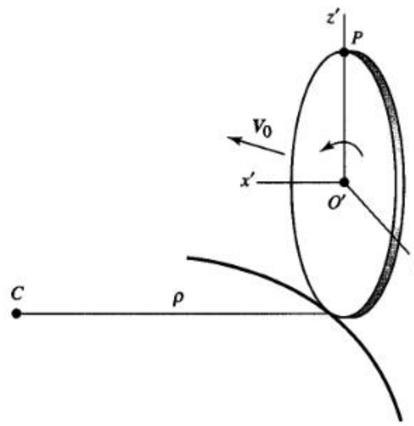


Figure 5.2.7 Wheel rolling on a curved track. The z'-axis remains vertical as the wheel turns.

We choose a coordinate system with $\hat{k}\,'$ always pointing up and $\hat{\imath}\,'$ always along the inner radius of the track. (Note the wheel is rotating with respect to this system).

This system rotates with $\vec{w} = (v_0/\rho)\hat{k}$.

$$\begin{cases} \sigma\,' \text{ does not coincide with } \sigma! \\ \text{Origin } \sigma\,' \text{ is accelerated: } \vec{A}_\sigma = \frac{v_0^2}{\rho}\hat{\imath}\,' \end{cases}$$

The acceleration of any point of the wheel viewed by the σ' system is v_0^2/b directed at the origin σ';

Therefore the top of the wheel: $\vec{a}_p\,' = -(v_0^2/b)\hat{k}\,'$
Also, the velocity of this point is $\vec{v}_p\,' = -v_0\hat{\jmath}\,'$.

So the coriolis acceleration is

$$2\vec{w} \times \vec{v}\,' \;=\; 2\left(\frac{v_0^2}{\rho}\right)\hat{k}\,' \times (-\hat{\jmath}\,') \;=\; \left(\frac{2v_0^2}{\rho}\right)\hat{\imath}\,'$$

Note: $\dot{\vec{w}} = 0$ and $\vec{w} \parallel \vec{r}\,'$, so transverse and centripetal are zero! Therefore, the net acceleration relative to the ground of the top of the wheel

$$\vec{a}_p \;=\; \vec{a_p}' + 2\vec{w} \times \vec{v}\,' + \vec{A}_\sigma \;=\; \frac{3v_0^2}{\rho}\hat{\imath}\,' - \frac{v_0^2}{b}\hat{k}\,'$$

Example 3: Bead on a rod; bead starts at a fixed point with initial velocity $\vec{v}_0\,' = wl\hat{i}$. When does it escape the rod?

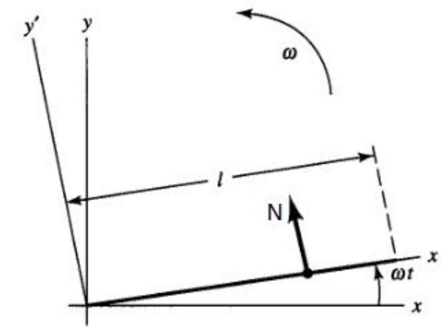


Figure 5.3.3 Bead sliding along a smooth rod rotating at constant angular velocity $\boldsymbol{\omega}$ about an axis fixed at one end.

We solve in the rotating reference frame:

$$\Sigma F' \;=\; N\hat{j}\,' - 2m\vec{w} \times \vec{v}\,' - m\vec{w} \times (\vec{w} \times \vec{r}\,') \;=\; m\ddot{\vec{r}}\,'$$

Note that $\vec{w} = w\hat{k}\,'$ and $\vec{v}\,' = \dot{x}\,'\hat{i}\,'$ and $\vec{r}\,' = x'\hat{i}\,'$ and $\ddot{\vec{r}}\,' = \ddot{x}'\hat{i}\,'$

$$N\hat{j}\,' - 2mw\dot{x}\,'\hat{j}\,' - mw^2x'(-\hat{i}\,') \;=\; m\ddot{x}\,'\hat{i}\,'$$

$$\implies \quad \begin{cases} N = 2mw\dot{x}\,' \quad , & \text{(Coriolis cancels normal force!)} \\ mw^2x' = m\ddot{x}\,' \quad , & \text{(Centrifugal force pulls particle to the outside!)} \end{cases}$$

$$\boxed{\ddot{x}\,' = w^2x'}$$

$$\begin{aligned} x'(t) &= Ae^{wt} + Be^{-wt} \\ \dot{x}'(t) &= w(Ae^{wt} - Be^{-wt}) \end{aligned}$$

$$x'(t=0) = 0 \implies A + B = 0 \implies \boxed{A = -B}$$

$$\dot{x}'(t=0) = v_0' \implies w(A-B) = {v_0}' \implies w(2A) = {v_0}' \implies \boxed{A = \frac{v_0'}{2w}} \quad , \quad \boxed{B = -\frac{v_0'}{2w}}$$

$$\implies \quad x'(t) \;=\; \frac{v_0'}{2w}\left(e^{wt} - e^{-wt}\right) \;=\; \frac{v_0'}{w}\sinh(wt)$$

Time to escape rod:

$$x'(t=T) \;=\; l \;=\; \frac{v_0'}{w}\sinh(wT)$$

$$\implies \quad T = \frac{1}{w}\sinh^{-1}\left(\frac{wl}{v_0'}\right) \;=\; \frac{1}{w}\sinh^{-1}(1) \;=\; \boxed{\frac{0.88}{w}}$$