Lecture 14: Motion of a Projectile on the Surface of the Earth

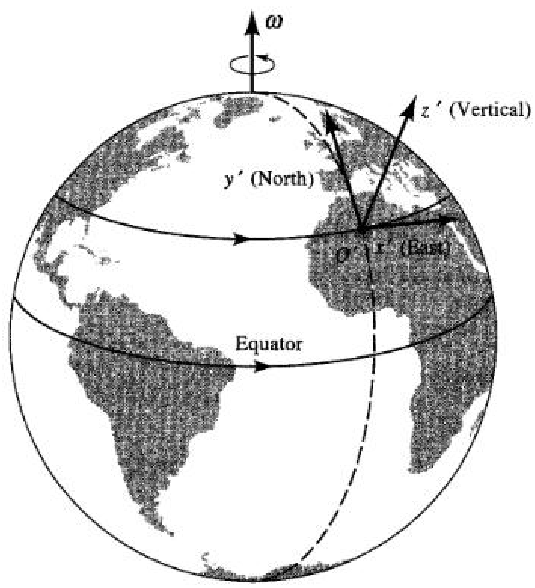
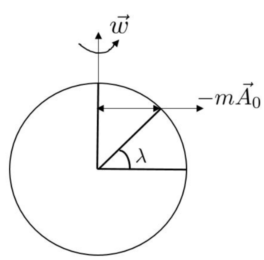


Figure 5.4.3 Coordinate axes for analyzing projectile motion.

$$\boxed{m\ddot{\vec{r}}\,' \;=\; \vec{F} \;+\; m\vec{g}_0 \;-\; m\vec{A}_0 \;-\; 2m\vec{w}\times\dot{\vec{r}}\,' \;-\; m\vec{w}\times(\vec{w}\times\vec{r}\,')}$$

The last term is the addition centrifugal force for $\vec{r}\,' \neq 0$ and is very small.



$-m\vec{A}_0$ is a centrifugal force where $|\vec{A}_0| = w^2 r_e \cos\lambda$. We redefine $\vec{g} = (\vec{g}_0 - \vec{A}_0) \equiv -g\hat{k}\,'$. That's what we mean by vertical.

In addition, we drop the terms that are $O(w^2)$ in comparison to $O(w)$ terms. Since $w_{earth} = 2\pi/(24 \times 3600s) = 7.3 \times 10^{-5}$rad/s, we have the following great approximation:

$$\boxed{m\ddot{\vec{r}}\,' \;=\; -mg\hat{k}\,' - 2m\vec{w} \times \dot{\vec{r}}\,'}$$

Since $\vec{w} = w[(\cos\lambda)\hat{j}' + (\sin\lambda)\hat{k}']$, it follows that

$$\vec{w} \times \dot{\vec{r}}\,' \;=\; \begin{vmatrix} \hat{i}\,' & \hat{j}\,' & \hat{k}\,' \\ 0 & w\cos(\lambda) & w\sin(\lambda) \\ \dot{x}\,' & \dot{y}\,' & \dot{z}\,' \end{vmatrix}$$

$$= \hat{i}\,'[w\cos(\lambda) - w\sin(\lambda)\dot{y}\,'] \quad + \quad \hat{j}\,'[w\sin(\lambda)\dot{x}\,'] \quad + \quad \hat{k}\,'[-w\cos(\lambda)\dot{x}\,']$$

We thus have that:

$$\begin{cases} \ddot{x}\,' \;=\; -2w[\dot{z}\,'\cos(\lambda) - \sin(\lambda)\dot{y}\,'] \\ \\ \ddot{y}\,' \;=\; -2w\sin(\lambda)\dot{x}\,' \\ \\ \ddot{z}\,' \;=\; 2w\cos(\lambda)\dot{x}\,' - g \end{cases}$$

$$\textbf{Integrate}: \begin{cases} \dot{x}\,' - \dot{x}_0\,' \;=\; -2w[(z' - z_0')\cos(\lambda) - \sin(\lambda)(y' - y_0')] \\ \\ \dot{y}\,' - \dot{y}_0\,' \;=\; -2w\sin(\lambda)(x' - x_0') \\ \\ \dot{z}\,' - \dot{z}_0\,' \;=\; 2w\cos(\lambda)(x' - x_0')\,' - gt \end{cases}$$

$$\implies \quad \ddot{x}\,' \;=\; -2w[\cos(\lambda)(\dot{z}_0'+2w\cos(\lambda)(x'-x_0')-gt) \quad - \quad \sin(\lambda)(\dot{y}_0'-2w\sin(\lambda)(x'-x_0'))]$$

Now removing the $O(w^2)$ terms...

$$\implies \quad \ddot{x}' \;=\; -\,2w[\cos(\lambda)\dot{z}_0' - \sin(\lambda)\dot{y}_0' - g\cos(\lambda)t]$$

We integrate again:

$$\dot{x} - \dot{x}_0 \;=\; -\,2w[\cos(\lambda)\dot{z}_0 - \sin(\lambda)\dot{y}_0']t + 2wg\cos(\lambda)\cdot\frac{t^2}{2}$$

We integrate yet again:

$$\begin{cases} x'(t) \;=\; x_0' \;+\; \dot{x}_0't \;-\; wt^2[\cos(\lambda)\dot{z}_0' - \sin(\lambda)\dot{y}_0'] \;+\; \frac{1}{3}wgt^3\cos(\lambda) \\ y'(t) = y_0' \;+\; \dot{y}_0't \;-\; w\dot{x}_0't^2\sin(\lambda) \\ z'(t) = z_0' \;+\; \dot{z}_0't \;-\; \frac{1}{2}gt^2 \;+\; w\dot{x}_0't^2\cos(\lambda) \end{cases}$$

Example 1: A falling Body

A body is dropped a height h above the ground. At $t = 0$, $\vec{v}\,' = 0$ and $\vec{r}\,' = h\hat{k}$.

The equations become

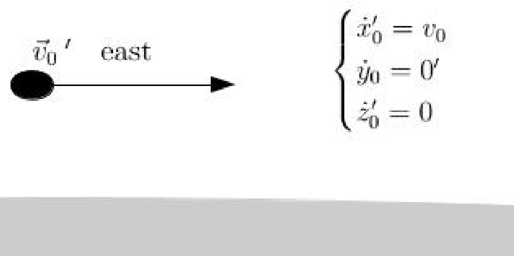
$$\begin{cases} x'(t) &= \frac{1}{3}wgt^3\cos(\lambda) \\ y'(t) &= 0 \\ z'(t) &= -\frac{1}{2}gt^2 + h \end{cases}$$

As it falls, it drifts to the east. It hits the ground at $T = \sqrt{2h/g}$

$$\implies \quad x'(T) = \frac{1}{3}wg\left(\frac{2h}{g}\right)^{3/2}\cos(\lambda) \;=\; \frac{1}{3}w\sqrt{\frac{8h^3}{g}}\cos(\lambda)$$

Plugging in $h = 100m$, $\lambda = 95°$ and $w = 7.3\times10^{-5}$rad/s yields $x'(T) = 1.55$cm.

Example 2: Deflection of a rifle bullet



$$\implies \quad y'(t) = -wv'_0 t^2 \sin(\lambda)$$

The projectile goes south (north hemisphere $\lambda > 0$) or north (south hemisphere $\lambda < 0$). Let R be the range of the projectile. Then

$$R \approx v_0 t_{flight} \quad \implies \quad |\Delta| \approx wv'_0 \frac{R^2}{v_0^2} \sin(\lambda) \;\; = \;\; \frac{wR^2}{v_0} |\sin(\lambda)|$$

This is the same for all directions, provided the projectiles $\vec{v}_0$ is in the xy plane (parallel to the ground). This can be shown by examining the Coriolis force:

$$\vec{w} \times \vec{v} = \hat{\imath}\,'(-w\dot{y}' \sin\lambda) + \hat{\jmath}\,'(w\dot{x}' \sin\lambda) + \hat{k}(-w\dot{x}' \cos\lambda)$$

Note that the $\hat{\imath}\,'$ and $\hat{\jmath}\,'$ components of the force have the same magnitude.