## Lecture 15: Gravitation and Central Forces

Newton's law of universal gravitation for point particles is

$$
\vec{F}_{i j}=G \frac{m_{i} m_{j}}{r_{i j}^{2}} \cdot \frac{\vec{r}_{i j}}{r_{i j}}
$$



What about the force between a uniform sphere of mass and a point particle?
Newton was bothered by this question, not sure whether or not his law would hold when the distance between objects was less than their radius. He invented integrals to deal with this problem (what a badass).

Consider a thin spherical shell:

Figure 6.2.1 Coordinates for calculating the gravitational field of a spherical shell.


$$
\Delta M_{\text {ring }}=\sigma \cdot 2 \pi(R \sin \theta)(R \Delta \theta)
$$

where $\sigma$ is the area density of the mass. By symmetry, only the component of $\Sigma_{P} \Delta \vec{F}_{Q}$ along $O P$ survives:

$$
\begin{aligned}
\Delta F & =G m \frac{\Delta M_{\text {ring }}}{s^{2}} \cos (\phi) \\
\Longrightarrow \quad d F & =G m \cdot 2 \pi \sigma R^{2} \cdot \frac{\sin \theta \cos \phi}{s^{2}} d \theta
\end{aligned}
$$

It follows that

$$
F=G m\left(2 \pi \sigma R^{2} \int_{0}^{\pi} d \theta \frac{\sin \theta \cos \phi}{s^{2}}\right.
$$

To evaluate the integral, let's express the integrand as a function of $s$.
Triangle $O P Q$, law of cosines for $\theta: \quad s^{2}+R^{2}-2 r R \cos \theta=s^{2}$. Taking $d / d \theta$ of both sides yields:

$$
2 r R \sin \theta=2 r \frac{d r}{d \theta} \Longrightarrow \sin \theta d \theta=\frac{s d s}{R r}
$$

Law of cosines for $\phi$ :

$$
\begin{gathered}
s^{2}+r^{2}-2 r s \cos \phi=R^{2} \Longrightarrow \cos \phi=\frac{s^{2}+r^{2}-R^{2}}{2 r s} \\
\Longrightarrow \quad F_{\text {shell }}=G m\left(2 \pi \sigma R^{2}\right) \int_{r-R}^{r+R} \frac{s^{2}+r^{2}+R^{2}}{2 r s} \frac{s d s}{R r} \frac{1}{s^{2}}
\end{gathered}
$$

$$
\begin{gathered}
F_{\text {shell }}=\frac{G m M_{\text {shell }}}{4 r^{2} R} \int_{r-R}^{r+R} d s\left[1+\frac{r^{2}-R^{2}}{s^{2}}\right] \\
F_{\text {shell }}=\frac{G m M_{\text {shell }}}{4 r^{2} R}\left(2 R-\left(r^{2}-R^{2}\right)\left[\frac{1}{s}\right]_{r-R}^{r+R}\right) \\
F_{\text {shell }}=\frac{G m M_{\text {shell }}}{4 r^{2} R} \cdot 4 R \Longrightarrow F_{\text {shell }}=\frac{G m M_{\text {shell }}}{r^{2}}
\end{gathered}
$$

## Kepler's Laws of Planetary Motion

Kepler's laws are empirical laws, deduced from detailed analysis of planetary motion. They can be derived from Newton's laws and gravity.
(i): Law of Ellipses: The orbit of each planet is an ellipse, with the sun located at one of its foci.
(ii): Law of Equal Areas: A line drawn between the sun and the planet sweeps out equal areas in equal times as the planet orbits the sun.
(iii): Harmonic Law: The square of the sidereal period of a planet (time it takes to orbit the sun relative to the stars) is directly proportional to the cube of the semi major axis of the planet's orbit.

The second law is identical to angular momentum, and requires only $\vec{F}=f(r) \hat{e}_{r} \quad$ (or $V=V(r)$ a central force)

Angular momentum is defined as $\vec{L}=\vec{r} \times \vec{p}$

$$
\frac{d \vec{L}}{d t}=\frac{d \vec{r}}{d t} \times \vec{p}+\vec{r} \times \frac{d \vec{p}}{d t}=\vec{r} \times \vec{F}=\vec{\tau}
$$

If $\vec{F}=L(r) \hat{e}_{r}$ then

$$
\frac{d \vec{L}}{d t}=0 \quad \Longrightarrow \quad\left\{\begin{array}{l}
\vec{r} \cdot \vec{L}=0 \\
\vec{v} \cdot \vec{L}=0
\end{array}\right.
$$

Both $\vec{r}$ and $\vec{v}$ are perpendicular to a constant vector $\Longrightarrow$ particle moves in a plane $\perp$ to the field vector $\vec{L}$.
$\Longrightarrow$ problem of a central field can be reduced to 2D always!

$$
\begin{gathered}
\vec{r}=r \hat{e}_{r} \quad, \quad \hat{e}_{r}=(\cos (\theta) \hat{i}+\sin (\theta) \hat{j}) \\
\dot{\vec{r}}=\vec{v}=\dot{r} \hat{e}_{r}+r\left(\dot{\hat{e}}_{r}\right)=\dot{r} \hat{e}_{r}+r(-\sin (\theta) \hat{i}+\cos (\theta) \hat{j}) \hat{\theta}=\dot{r} \hat{e}_{r}+r \dot{\theta} \hat{e}_{\theta}
\end{gathered}
$$

Note that $-\sin (\theta) \hat{i}+\cos (\theta) \hat{j}=\hat{e}_{\theta}$.

## Angular Momentum and Areal Velocity

Figure 6.4.1 (a) Angular momentum $L=|\mathbf{r} \times m \mathbf{v}|$ of a particle moving in a central field. (b) Area $d A=\frac{1}{2}|\mathbf{r} \times d \mathbf{r}|$ swept out by the radius vector $\mathbf{r}$ of the particle as it moves in a central field.

(a)

(b)

$$
\begin{gathered}
\vec{v}=\hat{e}_{r} \dot{r}+\hat{e}_{\theta} r \dot{\theta} \\
|\vec{L}|=|\vec{r} \times m \vec{v}|=m\left|r \hat{e}_{r} \times\left(\dot{r} \hat{e}_{r}+r \dot{\theta} \hat{e}_{\theta}\right)\right|=m r^{2} \dot{\theta}\left|\hat{e}_{r} \times \hat{e}_{\theta}\right|
\end{gathered}
$$

Note that $\hat{e}_{r} \times \hat{e}_{\theta}=-\hat{k}$ so we have that

$$
L=m r^{2} \dot{\theta}=\text { constant }
$$

## Calculating the Areal Velocity:

$$
d A=\frac{1}{2}|\vec{r} \times d \vec{r}|=\frac{1}{2}\left|r \hat{e}_{r} \times\left(d r \hat{e}_{r}+r d \theta \hat{e}_{\theta}\right)\right|=\frac{1}{2} r^{2} d \theta
$$

Note: Any increment of motion along $\hat{e}_{r}$ does not add to or subtract from the arc $d S$, nor $L$
Path for area to be swept:

$$
\begin{aligned}
\frac{d A}{d t} & =\frac{1}{2} r^{2} \dot{\theta}=\frac{L}{2 m}=\text { constant } \\
& \Longrightarrow A\left(t_{0}+T\right)-A\left(t_{0}\right)=\frac{L}{2 m} T
\end{aligned}
$$

Another Way:

$$
d A=\frac{1}{2}|\vec{r} \times d \vec{r}| \quad \Longrightarrow \quad \frac{d A}{d t}=\frac{1}{2}|\vec{r} \times d \vec{r} / d t|=\frac{1}{2 m}|\vec{r} \times \vec{p}|=\frac{L}{2 m}
$$

Example: Which $f(r)$ has all circular orbits with identical areal velocity $\dot{A}$ ?

$$
\begin{gathered}
\ddot{\vec{r}}=\frac{d}{d t}\left(\dot{r} \hat{e}_{r}+r \dot{\theta} \hat{e}_{\theta}\right)=\ddot{r} \hat{e}_{r}+\dot{r}\left(\dot{\hat{e}_{r}}\right)+\dot{r} \dot{\theta} e_{\theta}^{2}+r \ddot{\theta} \hat{e}_{\theta}+r \dot{\theta}\left(\dot{\hat{e}_{\theta}}\right) \\
=\left(\ddot{r}-r \dot{\theta}^{2}\right) \hat{e}_{r}+(2 \dot{r} \dot{\theta}+r \ddot{\theta}) \hat{e}_{\theta}
\end{gathered}
$$

In this case

$$
a_{r}=\left(\ddot{r}-r \hat{\theta}^{2}\right)=-r \dot{\theta}^{2}=a_{c}
$$

Now

$$
f(r)=-m r \dot{\theta}^{2}
$$

So we use

$$
\dot{A}=\frac{1}{2} r^{2} \dot{\theta} \quad \Longrightarrow \quad f(r)=-\frac{4 m}{r^{3}} \dot{A}^{2} \quad \Longrightarrow \quad f(r) \propto \frac{1}{r^{3}}
$$

Note that $\dot{A}$ for a $1 / r^{2}$ force does depend on radius r! For a circular orbit:

$$
-\frac{G m M}{r^{2}}=-m r \dot{\theta}^{2} \quad \Longrightarrow \quad \dot{\theta}^{2}=\frac{G M}{r^{3}} \quad \Longrightarrow \quad \dot{A}=\frac{1}{2} r^{2} \sqrt{\frac{G M}{r^{3}}}=\frac{\sqrt{G M}}{2} \sqrt{r}
$$

We find that $\dot{A}$ is larger when $r$ is larger.

