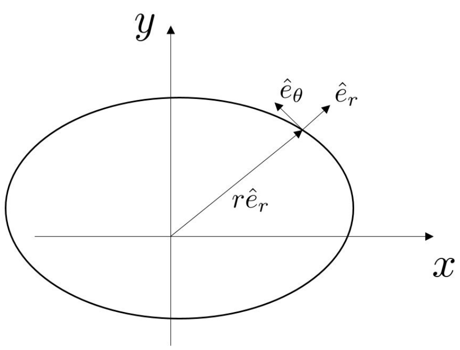
# Lecture 16: Kepler's First Law: The Law of Ellipses

**Central Force**: $\vec{F}(r) = f(r)\hat{e}_r$




$$\begin{cases} \hat{e}_r = (\cos\theta, \sin\theta) \\ \hat{e}_\theta = (-\sin\theta, \cos\theta) \end{cases}$$

Finding the equations of motion for $r$ and $\theta$:

$$m\ddot{\vec{r}} \;=\; m\frac{d}{dt}\frac{d}{dt}(r\hat{e}_r) \;=\; m\frac{d}{dt}(\dot{r}\hat{e}_r + r\dot{\hat{e}}_r) \;=\; m(\ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\hat{e}}_r + r\ddot{\hat{e}}_r)$$

We also have that

$$\begin{aligned} \dot{\hat{e}}_r &= (-\sin\theta, \cos\theta)\dot{\theta} \;=\; \dot{\theta}\hat{e}_\theta \\ \ddot{\hat{e}}_r &= \ddot{\theta}\hat{\theta} + \dot{\theta}\dot{\hat{\theta}} \;=\; \ddot{\theta}\hat{e}_\theta + \dot{\theta}(-\cos\theta, -\sin\theta)\dot{\theta} \\ &\boxed{\ddot{\hat{e}}_r \;=\; \ddot{\theta}\hat{e}_\theta - \dot{\theta}^2\hat{e}_r} \end{aligned}$$

So hence

$$m\ddot{\vec{r}} \;=\; m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta\right] \;=\; f(r)\hat{e}_r$$

$$\Longrightarrow \quad \begin{cases} m(\ddot{r} - r\dot{\theta}^2) \;=\; f(r)) \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) \;=\; 0 \end{cases}$$

The second equation implies that

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \Longrightarrow \quad r^2\dot{\theta} = \text{const} = l = \frac{L}{m} = |\vec{r}\times\vec{v}|$$

The quantity $|\vec{r}\times\vec{v}|$ is the angular momentum per unit mass. It follows that

$$\boxed{\dot{\theta} = \frac{l}{r^2}}$$

Lets try to get rid of time and find an equation that relates $r$ to $\theta$. To achieve this, consider the convenient change of variable $u = 1/r$.

$$r = \frac{1}{u} \quad \Longrightarrow \quad \dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{d\theta}{dt}\frac{du}{d\theta} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -l\frac{du}{d\theta}$$

$$\ddot{r} = -l\dot{\theta}\frac{d^2u}{d\theta^2} \quad = \quad -\,l^2u^2\frac{d^2u}{d\theta^2}$$

Plugging these into the equations of motion:

$$\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{m} \quad \Longrightarrow \quad -\,l^2u^2\frac{d^2u}{d\theta^2} - \frac{\dot{\theta}^2}{u} \quad = \quad \frac{f(1/u)}{m}$$

$$\Longrightarrow \quad -\,l^2u^2\frac{d^2u}{d\theta^2} - u^3l^2 = \frac{f(1/u)}{m}$$

$$\boxed{\frac{d^2u}{d\theta^2} + u \quad = \quad -\,\frac{f(1/u)}{ml^2u^2}}$$

This is very convenient for the orbit! It allows us to find $u = u(\theta)$ (or $r = r(\theta)$). Also, given $r = r(\theta)$ we can find the force field $f(r)$.

**Example:** A particle in a central field moves in the spiral orbit $r = c\theta^2$.
(a) Determine $f(r)$. (b) Find how $\theta$ depends on $t$

(a):

$$\implies \quad u = c^{-1}\theta^{-2} \quad \implies \quad \frac{-f(1/u)}{ml^2u^2} \;=\; \frac{d^2u}{d\theta^2} + u \;=\; 6c^{-1}\theta^{-4} + u$$

$$\implies \quad \frac{-f(1/u)}{ml^2u^2} \;=\; 6cu^2 + u \;=\; \frac{6c}{r^2} + \frac{1}{r}$$

$$\implies \quad \boxed{f(r) \;=\; -\,ml^2\left[\frac{6c}{r^4} + \frac{1}{r^3}\right]}$$

(b):

$$\dot{\theta} \;=\; lu^2 \;=\; lc^{-2}\theta^{-4}$$

$$\theta^4 d\theta = lc^{-2}dt \quad \implies \quad \frac{\theta^5}{5} = lc^{-2}t$$

$$\implies \quad \boxed{\theta = \left[\frac{5l}{c^2}t\right]^{1/5} \propto t^{1/5}}$$

## Inverse Square Law

Consider $f(r) = -k/r^2 \qquad (k = GmM)$. We assume $M >> m$ for now but later we will generalize.

$$\implies \quad \frac{d^2u}{d\theta^2} + u \;=\; \frac{-r^2 f(r)}{ml^2} \;=\; -\,\frac{r^2}{ml^2}\left(-\frac{k}{r^2}\right) \;=\; \frac{k}{ml^2}$$

$$\boxed{\frac{d^2u}{d\theta^2} + u \;=\; \frac{k}{ml^2}}$$

Just like the harmonic oscillator subject to a constant force, the solution is
$u(\theta) = A\cos(\theta - \theta_0) + k/ml^2$

$$\implies \quad \boxed{r = \frac{1}{\frac{k}{ml^2} + A\cos\theta}}$$

(Set $\theta_0 = 0$; assume $\theta$ is measured from distance of closest approach). This is an ellipse! To see this, write

$$\boxed{r = \frac{ml^2/k}{1 + \frac{Aml^2}{k}\cos\theta}}$$

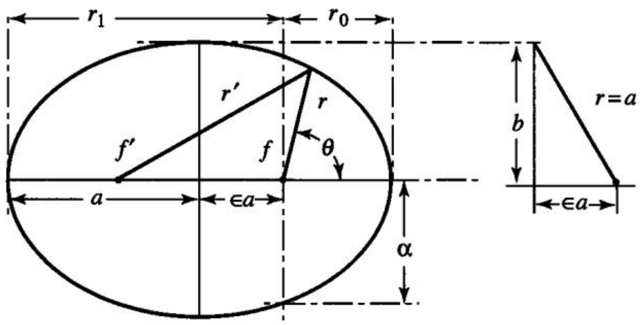


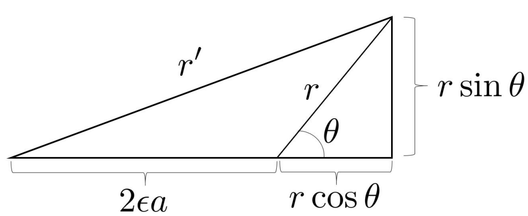

**Figure 6.5.1** The ellipse

- $f, f'$ The two foci of the ellipse
- $a$ Semimajor axis
- $b$ Semiminor axis: $b = (1-\epsilon^2)^{1/2} a$
- $\epsilon$ Eccentricity: each focus displaced from center by $\epsilon a$
- $\alpha$ Latus rectum: Distance of focus from point on the ellipse perpendicular to major axis: $\alpha = (1-\epsilon^2)\, a$
- $r_0$ Distance from the focus to the pericenter: $r_0 = (1-\epsilon)a$
- $r_1$ Distance from the focus to the apocenter: $r_1 = (1+\epsilon)a$

**Geometrical Definition of Ellipse**: Locus of all points whose sum of distances from two foci is constant:

$$r + r' = \text{const} = (1-a)a + (1+a)a = 2a$$

Let's show that this property implies $r(\theta)$ like above




Pythagoras: $r'^2 = (r\sin\theta)^2 + (2\epsilon a + r\cos\theta)^2$

Use $r' = 2a - r$:

$$\begin{aligned} (2a-r)^2 &= r^2\sin^2\theta + 4\epsilon^2 a^2 + r^2\cos^2\theta + 4\epsilon a r\cos\theta \\ 4a^2 - 4ar + r^2 &= r^2 + 4\epsilon^2 a^2 + 4\epsilon a r\cos\theta \\ a - r &= \epsilon^2 a + \epsilon r\cos\theta \end{aligned}$$

$$(1-\epsilon^2)a = r(1+\epsilon\cos\theta) \implies \boxed{r = \frac{(1-\epsilon^2)a}{1+\epsilon\cos\theta}}$$

$r = \alpha$ at $\theta = \pi/2$, so

$$\boxed{r = \frac{\alpha}{1+\epsilon\cos\theta}} \implies \begin{cases} \alpha = \frac{ml^2}{k} \\ \epsilon = \frac{ml^2}{k}A \end{cases}$$

**Note**:

$$r_0 = r(\theta = 0) = \frac{\alpha}{1+\epsilon}$$
$$r_1 = r(\theta = \pi) = \frac{\alpha}{1-\epsilon}$$

Actually, the equation $r = 1/(1+\epsilon\cos\theta)$ does not describe any ellipse! The orbit can be:

1. $\epsilon = 0$ circle
2. $0 < \epsilon < 1$ ellipse
3. $\epsilon = 1$ parabola
4. $\epsilon > 1$ hyperbola

These are conic sections: