Lecture 17: Kepler's Third Law

$$r(\theta) = \frac{\alpha}{1+\epsilon\cos\theta} \qquad \begin{cases} \alpha = ml^2/k \\ \epsilon = \frac{ml^2}{k}A \end{cases}$$

Example 1: Calculate the speed of a satellite in circular orbit about Earth.

$$\begin{aligned} &\epsilon = 0 \qquad (A=0) \\ &k = GM_E m_s \\ &l = |\vec{r}\times\vec{v}| = r_c v_c \end{aligned}$$

Now from $r(\theta) = r_c$:

$$r_c = \frac{\frac{m_s(r_c v_c)^2}{GM_E m_s}}{1} = \frac{r_c^2 v_c^2}{GM_E} \implies v_c^2 = \frac{GM_E}{r_c} = \left(\frac{GM_E}{R_E^2}\right)\frac{R_E^2}{r_c} = \frac{gR_E^2}{r_c}$$

$$\boxed{v_c = \left(\frac{gR_E^2}{r_c}\right)^{1/2}}$$

For a low lying orbit, $r_c \approx R_E \implies v_c = \sqrt{gR_E} \approx 7920$ m/s ≈ 8 km/s.

Example 2: A spacecraft is at a low lying orbit at the earth with radius $r_c \approx R_E$. The most energy efficient way to send this spacecraft to the moon is to boost its speed when it's in circular orbit so that its orbit becomes an ellipse with perigee at r_c and apogee at $R_{moon} \approx 60R_E$. What is the required speed boost at perigee? On the diagram below, $r_0 = R_E$ and $r_1 = R_{moon}$.

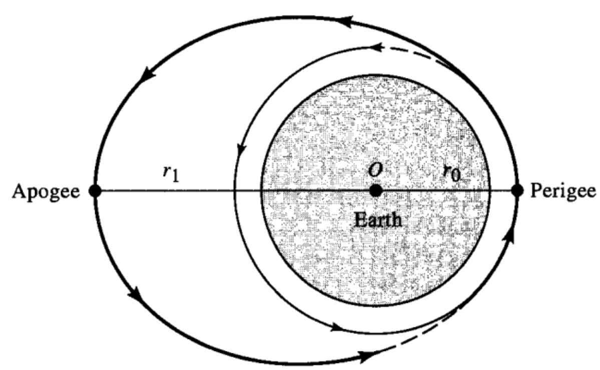


Figure 6.5.3 Spacecraft changing from a circular to an elliptical orbit.

At circular orbit,

$$r_c = \frac{ml^2}{k} = \frac{m(v_c r_c)^2}{k} \quad \Longrightarrow \quad r_c = \frac{k}{mv_c^2}$$

The new orbit must have new α_n and ϵ_n such that

$$r_n(\theta) = \frac{\alpha_n}{1 + \epsilon_n \cos\theta} \quad \Longrightarrow \quad \begin{cases} r_n(\theta = 0) = \frac{\alpha_n}{1+\epsilon_n} = r_c \quad \text{(perigee)} \\ r_n(\theta = \pi) = \frac{\alpha_n}{1-\epsilon_n} = R_{moon} \quad \text{(apogee)} \end{cases}$$

$$\text{Find } \alpha_n : \begin{cases} \alpha_n = r_c(1 + \epsilon_n) \\ \frac{r_c}{R_{moon}}\alpha_n = r_c(1 - \epsilon_n) \end{cases} \quad \Longrightarrow \quad \left(1 + \frac{r_c}{R_{moon}}\right)\alpha_n = 2r_c \quad \Longrightarrow \quad \alpha_n = \frac{2r_c}{1 + r_c/R_{moon}}$$

We also know that $\alpha_n = \frac{m}{k}{l_n}^2 = \frac{m}{k}(v_n r_c)^2$. Hence we equate:

$$\frac{m}{k}(v_n r_c)^2 = \frac{2r_c}{1 + r_c/R_{moon}}$$

$$\frac{m}{k}v_n^2 r_c = \frac{2}{1 + r_c/R_{moon}}$$

$$\frac{m}{k}v_n^2\left(\frac{k}{mv_c^2}\right) = \frac{2}{1 + r_c/R_{moon}} \quad \Longrightarrow \quad \left(\frac{v_n}{v_c}\right)^2 = \frac{2}{1 + r_c/R_{moon}} = \frac{2R_{moon}}{R_{moon} + r_c}$$

$$\left(\frac{v_n}{v_c}\right) = \sqrt{\frac{2 \times 60R_E}{61R_E}} = \boxed{1.40}$$

This is a 40% boost to a speed of 1.4×8 km/s $=$ $\boxed{11.2\text{km/s}}$

Kepler's Third Law:

$$\boxed{\tau^2 \propto a^3}$$

where τ is the period of orbit and a is distance from the sun. This was regarded as the universal relationship between period and distance: "the magic of the heavens."

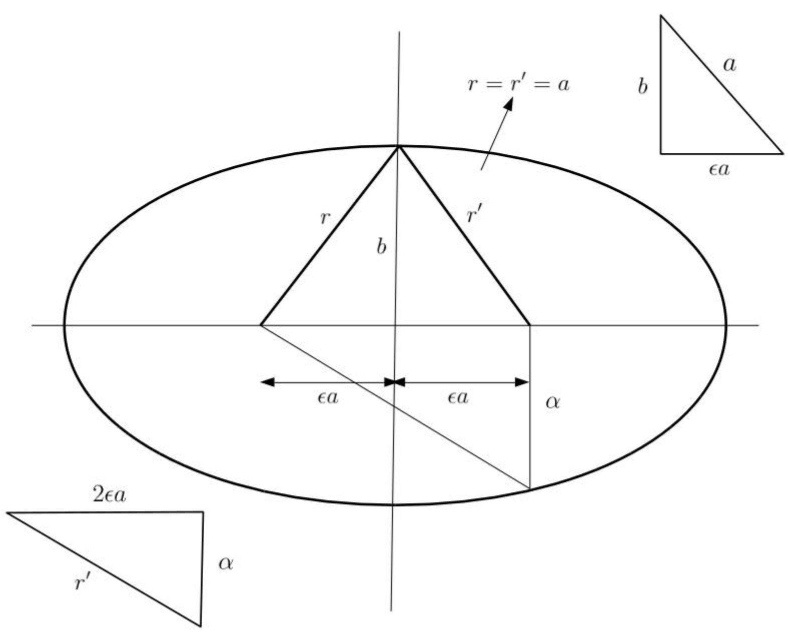
Let's start from the second law:

$$\dot{A} = \frac{L}{2m} = \frac{l}{2}$$

$$\int_0^\tau \dot{A}dt = \frac{l}{2}\tau \quad \Longrightarrow \quad A = \frac{l}{2}\tau \quad \Longrightarrow \quad \tau = \frac{2A}{l}$$

$$\text{But } A = \pi ab \quad \Longrightarrow \quad \tau = \frac{2\pi ab}{l}$$

We now take a brief break and show that $b = a\sqrt{1-\epsilon^2}$ and $\alpha = a(1-\epsilon^2)$. Note that on the diagram below, $r + r' = 2a$:



By the Pythagorean theorem we have that $b^2 + \epsilon^2 a^2 = a^2$ or equivalently that $b = a\sqrt{1-\epsilon^2}$. To show that $\alpha = a(1-\epsilon^2)$, we first note that $r' + \alpha = 2a \implies r' = 2a - \alpha$. Again, by the Pythagorean theorem,

$$r'^2 = (2\epsilon a)^2 + \alpha^2 \quad \Longrightarrow \quad (2a-\alpha)^2 = (2\epsilon a)^2 + \alpha^2$$

$$\Longrightarrow \quad \alpha = a(1-\epsilon^2)$$

Now back to the problem at hand:

$$\tau = \frac{2\pi a(a(1-\epsilon^2)^{1/2})}{l} \quad = \quad \frac{2\pi a^2}{l}\sqrt{1-\epsilon^2}$$

$$\implies \quad \tau^2 = \frac{4\pi^2 a^4}{l^2}(1-\epsilon^2)$$

$$\tau^2 = \frac{4\pi^2 a^4}{l^2}\frac{\alpha}{a} \quad = \quad 4\pi^2\left(\frac{\alpha}{l^2}\right)a^3$$

We insert $\alpha = ml^2/k$ and $k = GmM$ to get:

$$\tau^2 = 4\pi^2\left(\frac{ml^2}{GmMl^2}\right)a^3$$

$$\boxed{\tau^2 = \left(\frac{4\pi^2}{GM}\right)a^3}$$

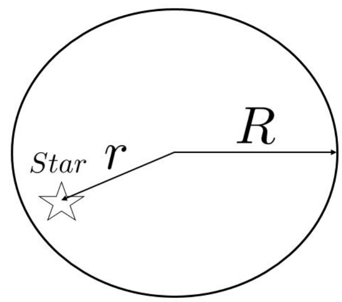
This is the same for all planets since the mass of the sun M is constant! If distances are measured in astronomical units 1AU$= a_{earth} = 1.50 \times 10^8$ km and periods expressed in earth years then $\tau^2 = a^3 \quad \implies \quad (4\pi^2/GM) = 1$

	Period		***Semimajor Cube***		***Eccentricity***
Planet	τ***(yr)***	**Square** $\tau^2(yr^2)$	***Axis*** a**(AU)**	a^3**(AU^3)**	ϵ
Mercury	0.241	**0.0581**	0.387	**0.0580**	0.206
Venus	0.615	**0.378**	0.723	**0.378**	0.007
Earth	1.000	**1.000**	1.000	**1.000**	0.017
Mars	1.881	**3.538**	1.524	**3.540**	0.093
Jupiter	11.86	**140.7**	5.203	**140.8**	0.048
Saturn	29.46	**867.9**	9.539	**868.0**	0.056
Uranus	84.01	**7058.**	19.18	**7056.**	0.047
Neptune	164.8	**27160.**	30.06	**27160.**	0.009
Pluto	247.7	**61360.**	39.440	**61350.**	0.249

An eccentricity near 0 implies a nearly circular orbit.

Dark Matter

What is the rotational speed of stars in a galaxy? Consider the simple galaxy model where the galaxy has uniform mass density $\rho = M/\frac{4}{3}\pi R^3$:



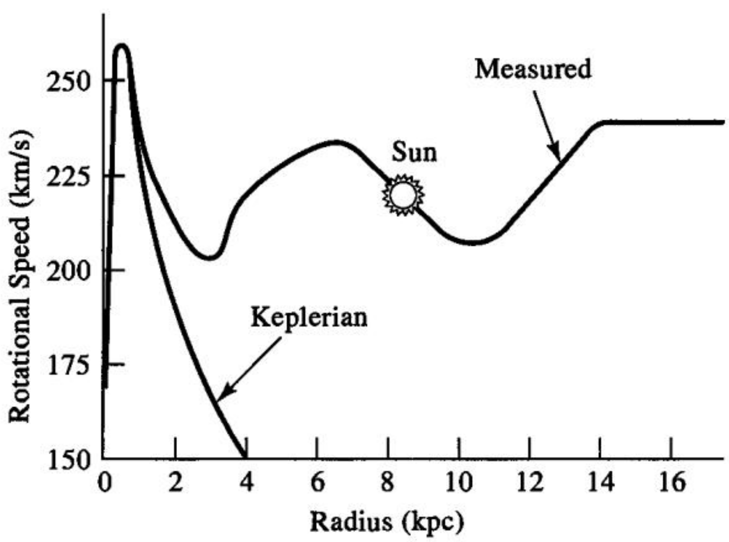
$$\frac{GMm}{r^2} = \frac{mv^2}{r} \quad \Longrightarrow \quad v = \frac{GM}{r} = \frac{G}{r}\rho\frac{4}{3}\pi r^3$$

$$\Longrightarrow \quad v^2 = \frac{4}{3}\pi G \frac{M_{gal}}{(4/3)\pi R^3} r^2 \quad = \quad \frac{GM_{gal}}{R^3} r^2$$

$$\Longrightarrow \quad v = \sqrt{\frac{GM_{gal}}{R^3}} r \quad \propto r$$

Hence the rotational speed for stars with $r < R$ is $\propto r$! For stars in the spiral arms of the galaxy, i.e $r > R$:

$$\frac{GM_{gal}m}{r^2} = \frac{mv^2}{r} \quad \Longrightarrow \quad v = \sqrt{\frac{GM_{gal}}{r}} \quad \propto \frac{1}{\sqrt{r}}$$



This suggests additional “dark matter” spread throughout the galaxy. Dark matter seems to make up approximately 75% of the universe (i.e., we only see 25% of matter required to account for gravitational motion.