Lecture 18: Potential Energy in a Gravitational Field

Are all central forces conservative? A central force is given by $\vec{F} = f(r)\hat{e}_r$.

$$\nabla \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta r & \hat{e}_\phi r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & rF_\phi \sin\theta \end{vmatrix}$$

Since the force is central, $F_\theta, F_\phi = 0$ and hence

$$\nabla \times \vec{F} = \frac{1}{r^2 \sin\theta}\left[r\hat{e}_\theta \frac{\partial}{\partial\phi} f(r) - r\sin\theta \hat{e}_\phi \frac{\partial}{\partial\theta} f(r)\right] = 0$$

So yes, all central forces are conservative.

$$\implies \quad \vec{F} = -\vec{\nabla} V(r) \quad , \quad f(r) = -\frac{dV}{dr}$$

$$V(r) = -\int_{\vec{r}_i}^{\vec{r}} \vec{F}\cdot d\vec{r} = -\int_{r_i}^{r} f(r)dr$$

Since $f(r) = -k/r^2$ we have that

$$V(r) = k\int_{r_i}^{r} \frac{dr}{r^2} = k\left(-\frac{1}{r}\right)_{r_i}^{r} = -k\left(\frac{1}{r} - \frac{1}{r_i}\right)$$

We choose $r_i = \infty$ so that we can define the **gravitational potential energy** as

$$\boxed{V(r) = -\frac{GMm}{r}}$$

In addition, we define the gravitational potential Φ as

$$\Phi(\vec{r}) = \lim_{m\to 0}\left[\frac{V(r)}{m}\right] = -\frac{GM}{r}$$

If we have a number of particles of mass $m_1, m_2, ..., m_i, ...$ located at $\vec{r}_1, \vec{r}_2, ...$ then the gravitational potential is

$$\Phi(\vec{r}) = \sum_i \Phi_i = -G\sum_i \frac{m_i}{|\vec{r}-\vec{r}_i|}$$

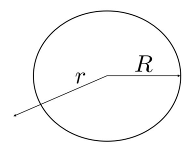
Because Φ is a scalar, it is generally easier to computer Φ as opposed to $\vec{F} = \sum_i \vec{F}_i$.

If the gravitational force is denoted by $\vec{F} = m\vec{g}$ we have

$$\boxed{\vec{F} = -\vec{\nabla}V} \qquad \boxed{\vec{g} = -\vec{\nabla}\Phi}$$

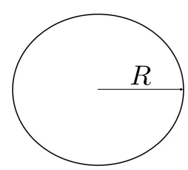
$\vec{g}$ is the "local field intensity", or the acceleration of gravity.

Integrate: Spherical Shell



$$\Phi(r) = -\frac{GM}{r} \quad \text{for } r > R$$
$$\Phi(r) = -\frac{GM}{R} \quad \text{constant for } r < R$$

Integrate: Mass Ring



$$\Phi(r) = -\frac{GM}{r}\left(1 + \frac{R^2}{4r^2} + ...\right) \quad r > R \quad \left(\approx -\frac{GM}{r} \text{ for } r >> R\right)$$

$$\Phi(r) = -\frac{GM}{r}\left(1 + \frac{r^2}{4R^2} + ...\right) \quad r < R$$

For $r < R$ this means that

$$\vec{g} = -\frac{\partial\Phi}{\partial r}\hat{e}_r = \frac{GM}{R}\frac{r}{2R^2}\hat{e}_r = \frac{GM}{2R^3}r\hat{e}_r \qquad \boxed{\text{Repulsive!}}$$

Energy Equation of an Orbit

$$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$
$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

and since $(1/2)mv^2 + V(r) = E$ =constant, it follows that

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = E$$

and now

$$u = \frac{1}{r} \quad , \quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} \quad \Longrightarrow \quad \boxed{-r^2\dot{\theta}\frac{du}{d\theta} = \dot{r}}$$

and since $-r^2\dot{\theta} = l$ we have that

$$\dot{r} = -l\frac{du}{d\theta}$$

$$l^2 = r^4\dot{\theta}^2 \quad \Longrightarrow \quad u^2l^2 = r^2\dot{\theta}^2$$

$$\boxed{\frac{1}{2}ml^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V\left(\frac{1}{u}\right) = E}$$

Inverse Square Field: Find the orbit parameters as a function of E.

$$V(r) = -\frac{k}{r} = -ku$$

$$\Longrightarrow \quad \frac{1}{2}ml^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku = E$$

$$\left(\frac{du}{d\theta}\right)^2 = \left(\frac{2E}{ml^2}\right) + \left(\frac{2k}{ml^2}\right)u - u^2 \quad \Longrightarrow \quad \frac{du}{\sqrt{-u^2 + \left(\frac{2k}{ml^2}\right)u + \left(\frac{2E}{ml^2}\right)}} = d\theta$$

Define $a = -1$ and $b = 2k/ml^2$ and $c = 2E/ml^2$.

$$\int \frac{du}{\sqrt{au^2 + b + c}} \;=\; \int_{\theta_0}^{\theta} d\theta \quad , \quad \text{use a table of integrals:}$$

$$\implies \quad (\theta - \theta_0) \;=\; \frac{1}{\sqrt{-a}} \cos^{-1}\left(-\frac{b + 2au}{\sqrt{b^2 - 4ac}}\right)$$

$$\implies \quad \cos\left[\sqrt{-a}(\theta - \theta_0)\right] \;=\; -\frac{b + 2au}{\sqrt{b^2 - 4ac}}$$

$$\implies \quad u = \frac{\sqrt{b^2 - 4ac}}{-2a} \cos\left[\sqrt{-a}(\theta - \theta_0)\right] + \frac{b}{-2a}$$

and now substituting in our values,

$$\implies \quad \frac{1}{r} = \frac{\sqrt{4\left(\frac{k}{ml^2}\right)^2 + \left(\frac{8E}{ml^2}\right)}}{2} \cos(\theta - \theta_0) + \frac{k}{ml^2}$$

$$\implies \quad \frac{1}{r} = \frac{k}{ml^2}\left[\sqrt{1 + \frac{2ml^2}{k^2} E} \cos(\theta - \theta_0) + 1\right]$$

$$\implies \quad \boxed{r = \frac{ml^2/k}{1 + \sqrt{1 + \frac{2ml^2}{k^2}} \cos(\theta - \theta_0)}}$$

Relationship Between Eccentricity and Energy

Recall that

$$\epsilon = \sqrt{1 + \frac{2ml^2}{k^2}E}$$

$$\begin{aligned} E < 0 &\implies \epsilon < 1 \quad \text{Elliptic} \\ E = 0 &\implies \epsilon = 1 \quad \text{Parabola} \\ E > 0 &\implies \epsilon > 1 \quad \text{Hyperbola} \end{aligned}$$

We know that $ml^2/k = \alpha = (1-\epsilon^2)a$ and so

$$\epsilon^2 = 1 + \frac{2E}{k}(1-\epsilon^2)a$$

$$\implies \quad 0 = (1-\epsilon^2) + \frac{2Ea}{k}(1-\epsilon^2) \quad \implies \quad \frac{2Ea}{k} = -1$$

$$\implies \quad \boxed{E = -\frac{k}{2a}}$$

This gives energy as a function of the semi-major axis.