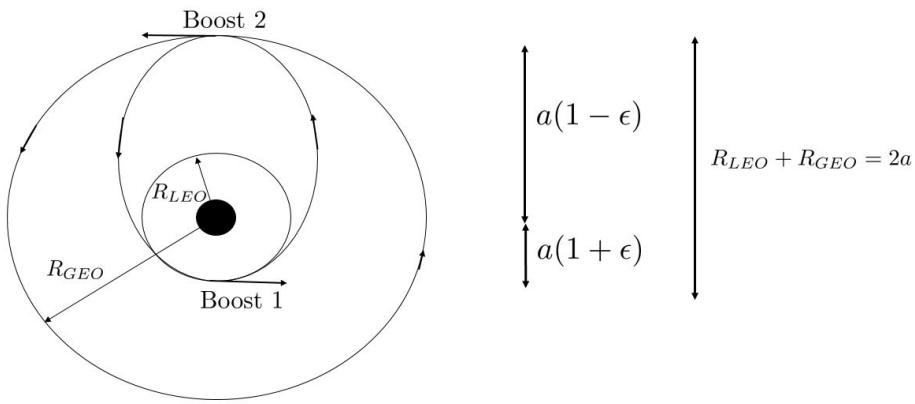
Lecture 19: Energy, Orbits, Radial Motion

Example: Placing a satellite into a geosynchronous orbit with *two velocity boosts*.



The satellite starts at low earth orbit (LEO) ($v \approx 26000$ km/h , $R_{LEO} \approx R_{earth}$). The first boost takes the satellite to an intermediate elliptical orbit:

$$\text{Note: } \begin{cases} R_{LEO} = a(1-\epsilon) \\ R_{GEO} = a(1+\epsilon) \end{cases}$$

where a is the semi major axis of the intermediate elliptical orbit. The energy of the elliptical orbit is given by

$$E = -\frac{k}{2a} = -\frac{GM_E m}{2a} = \frac{1}{2}mv_p^2 - \frac{GM_e m}{a(1-\epsilon)}$$

the value after the last equality represents the energy at the perigee.

$$\Longrightarrow \quad \frac{1}{2}mv_p^2 = \frac{GM_e m}{a}\left[\frac{1}{1-\epsilon} - \frac{1}{2}\right]$$

$$\Longrightarrow \quad v_p^2 = \frac{GM_E}{a}\left[\frac{2-(1-\epsilon)}{1-\epsilon}\right] \quad = \quad \frac{GM_E}{a}\left(\frac{1+\epsilon}{1-\epsilon}\right)$$

Substituting in R_{GEO} and R_{LEO} yields

$$v_p^2 = \frac{2GM_E}{R_{LEO} + R_{GEO}} \left(\frac{R_{GEO}}{R_{LEO}} \right)$$

This is the velocity at the perigee. The LEO velocity is given by

$$\frac{mv_{LEO}^2}{R_{LEO}} = \frac{GM_E m}{R_{LEO}^2} \implies v_{LEO}^2 = \frac{GM_E}{R_{LEO}}$$

First Boost:

$$\Delta v_1 = v_p - v_{LEO} \;=\; \sqrt{\frac{GM_E}{R_{LEO}}} \left[\sqrt{\frac{2R_{GEO}}{R_{GEO} + R_{LEO}}} - 1 \right]$$

To solve for R_{GEO}, we use the fact that $v_{GEO} = 2\pi R_{GEO}/T_E$ and $mv_{GEO}^2/R_{GEO} = GmM_E/R_{GEO}^2$.

$$\implies \quad R_{GEO} = \frac{GM_E}{v_{GEO}^2} = \frac{GM_E}{4\pi^2 R_{GEO}^2/T_E^2} \quad \implies \quad R_{GEO}^3 = \frac{GM_E T_E^2}{4\pi^2}$$

Hence $R_{GEO} = 42400$ km. Note that $M_E = 5.97 \times 10^{24}$ kg, $R_{LEO} = 6693$ km, $R_{GEO} = 42400$ km and hence it follows that $\Delta v_1 = 8600$ km/h.

Second Boost :

The energy at the apogee of the elliptical orbit is given by

$$E = -\frac{GM_E m}{2a} = \frac{1}{2} m v_a^2 - \frac{GM_E m}{a(1+\epsilon)}$$

$$\implies \quad v_a^2 = \frac{GM_E}{a} \left[\frac{2}{1+\epsilon} - 1 \right] = \frac{2GM_E}{(R_{LEO} + R_{GEO})} \left[\frac{1-\epsilon}{1+\epsilon} \right] \qquad \text{(apogee)}$$

$$v_a^2 = \frac{2GM_E}{(R_{LEO} + R_{GEO})} \left(\frac{R_{LEO}}{R_{GEO}} \right)$$

v_{GEO} is given by

$$\frac{mv_{GEO}^2}{R_{GEO}} = \frac{GM_E m}{R_{GEO}^2} \quad \Longrightarrow \quad v_{GEO}^2 = \frac{GM_E}{R_{GEO}}$$

Hence

$$\Delta v_2 = v_{GEO} - v_a \;=\; \sqrt{\frac{GM_E}{R_{GEO}}}\left[1 - \sqrt{\frac{2R_{LEO}}{R_{LEO}+R_{GEO}}}\right] \approx 5269\text{km/h}$$

We see that $\Delta v_1 + \Delta v_2 = 13869$ km/h, which is much less than $v_{LEO} \approx 26000$ km/h. (About 50% off).

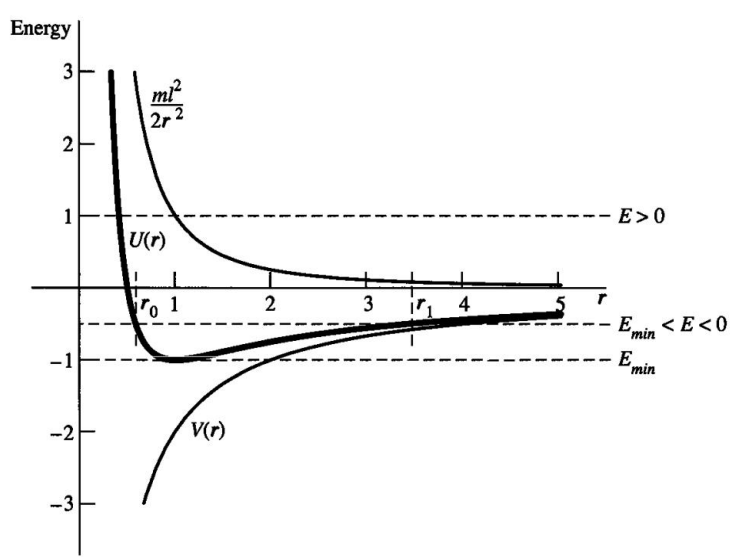
Effective Potential in Radial Motion

$$\begin{cases} 1/2mv^2 + V(r) = E \\ v^2 = \dot{r}^2 + (r\dot{\theta})^2 \\ l = |\vec{r}\times\vec{v}| = rr\dot{\theta} = r^2\dot{\theta} \end{cases} \quad \Longrightarrow \quad \frac{1}{2}m\left[\dot{r}^2 + \frac{l^2}{r^2}\right] + V(r) = E$$

$$\frac{1}{2}m\dot{r}^2 + \left[\frac{ml^2}{2r^2} + V(r)\right] = E$$

We call the term in the square brackets the effective potential ($U(r)$) or the "centrifugal potential."

Example: Suppose $V(r) = -k/r$. Then $U(r) = (ml^2/2)(1/r^2) - (k/r)$.



If $E < 0$, bounded r $\implies$ Ellipse! Turning points at $\dot{r} = 0$ or

$$U(r) = E(< 0) \quad \implies \quad \frac{ml^2}{2}\frac{1}{r^2} - \frac{k}{r} = E \quad \implies \quad Er^2 + kr - \frac{ml^2}{2} = 0$$

$$\implies \quad r_\pm = \frac{1}{2E}\left(-K \pm \sqrt{K^2 + \frac{4Eml^2}{2}}\right) \quad = \quad \frac{K \pm \sqrt{K^2 + 2Eml^2}}{-2E}$$

$$\implies \quad r_\pm = \frac{K \pm \sqrt{K^2 - 2|E|ml^2}}{2|E|}$$

(+ is the apogee and - is the perigee)

If we make the argument under the square root zero, the orbit is a circle. Now we have

$$E = E_{min} = -\frac{K^2}{2ml^2} \quad \implies \quad r_0 = \frac{K}{2|E_{min}|} = \frac{ml^2}{K}$$

Now using $l = r_0 v_0$ we have

$$r_0 = \frac{m}{K} r_0^2 v_0^2 \quad \implies \quad \frac{K}{r_0^2} = \frac{mv_0^2}{r_0}$$

Rotate effective U and find orbits: