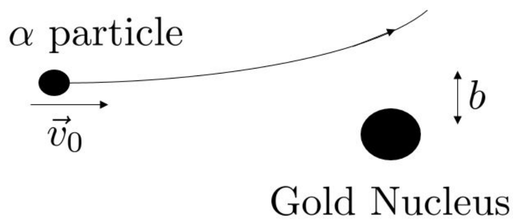
Lecture 20: Motion in an Inverse Square Repulsive Potential



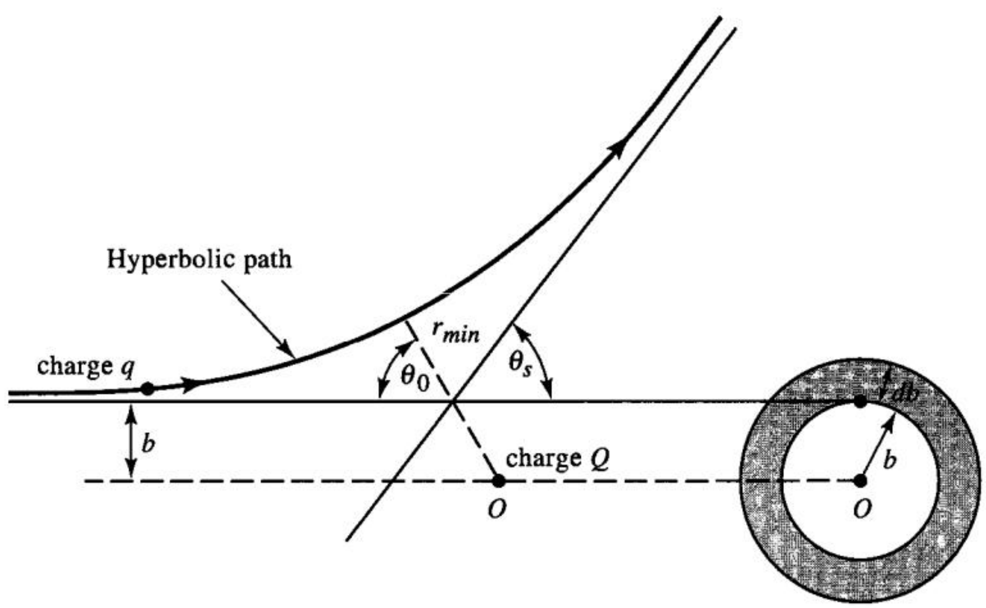
$$f(r) = \frac{Qq}{r^2} \quad (k = -Qq)$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(r)}{mu^2l^2} = -\frac{Qq}{ml^2}$$

$$\implies \quad u^{-1} = r(\theta) = \frac{1}{A\cos(\theta-\theta_0) - Qq/ml^2} \quad , \quad \text{or}$$

$$r(\theta) = \frac{ml^2Q^{-1}q^{-1}}{-1 + \sqrt{1 + \frac{2Eml^2}{Q^2q^2}}\cos(\theta-\theta_0)}$$

Since $E = +\frac{1}{2}mv_0^2$, the orbit is a hyperbola



We call b the impact parameter. We choose θ_0 such that $r(\theta = 0) = \infty$

$$\implies \boxed{\cos(\theta_0) = \frac{1}{\sqrt{1 + \frac{2Eml^2}{q^2Q^2}}}}$$

In addition, $r(\theta = \theta_0) = r_{min}$ and $r(\theta = 2\theta_0) = \infty$.

Since

$$\sin(\theta_0) = \sqrt{1 - \frac{1}{1 + \frac{2Eml^2}{q^2Q^2}}} = \frac{\sqrt{\frac{2Eml^2}{q^2Q^2}}}{\sqrt{1 + \frac{2Eml^2}{q^2Q^2}}}$$

we have that

$$\tan(\theta_0) = \sqrt{2mE}\frac{l}{Qq} = \tan\left(\frac{\pi}{2} - \frac{\theta_s}{2}\right) = \cot\left(\frac{\theta_s}{2}\right)$$

Recall that $l = |\vec{r} \times \vec{v}| = bv_0$

$$\implies \quad \cot\left(\frac{\theta_s}{2}\right) = \sqrt{2m\frac{1}{2}mv_0^2}\frac{bv_0}{Qq} = \frac{mv_0^2 b}{Qq} = \frac{2bE}{qQ}$$

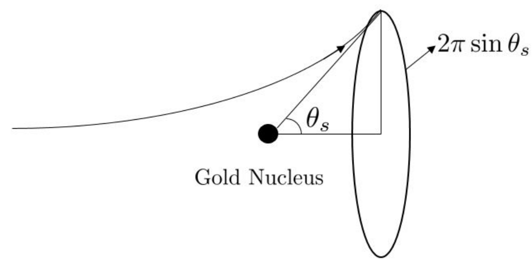
Differential Scattering Cross Section

$$dN = I\sigma(\theta_s)d\Omega$$

dN: # of scattered particles into $d\Omega$ in unit time
I: # of incident particles per unit area per unit time (flux)
σ: cross section area of each scattering center
Ω: scattered solid angle

$$I(2\pi b)db = I\sigma(\theta_s)2\pi\sin(\theta_s)d\theta_s \quad \implies \quad \boxed{\sigma(\theta_s) = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right|}$$

$I(2\pi b)db$ corresponds to the dN incident at $[b, b + db]$ and $I\sigma(\theta_s)2\pi\sin(\theta_s)d\theta_s$ corresponds to the scattered.



$$b = \frac{qQ}{2E}\cot\left(\frac{\theta_s}{2}\right) \quad \Longrightarrow \quad \frac{db}{d\theta_s} = \frac{qQ}{2E}\frac{(-1)}{2\sin^2(\theta_s/2)}$$

$$\Longrightarrow \quad \sigma(\theta_s) = \left(\frac{b}{\sin(\theta_s)}\right)\left(\frac{qQ}{2E}\right)\left(\frac{1}{\sin^2(\theta_s/2)}\right)$$

Note that $\sin(\theta_s) = 2\sin(\theta_s/2)\cos(\theta_s/2)$. In addition, we know that $b = (Qq/mv_0^2)\cos(\theta_s/2)$ and hence

$$\sigma(\theta_s) = \left(\frac{(Qq)^2}{(2E)^2}\right)\left(\frac{\cos(\theta_s/2)}{\sin(\theta_s/2)}\right)\left(\frac{1}{2\sin^2(\theta_s/2)}\right)\left(\frac{1}{2\sin(\theta_s/2)\cos(\theta_s/2)}\right)$$

$$\Longrightarrow \quad \boxed{\sigma(\theta_s) = \frac{(Qq)^2}{16E^2}\cdot\frac{1}{\sin^4(\theta_s/2)}}$$

Rutherford scattering cross sections are very different from the Thomson jelly model.

Some particles are deflected by $\theta_s = \pi$.

PS: quantum mechanics gives identical results!