Lecture 21: Stability of Nearly Circular Orbits

$$\frac{1}{2}m\dot{r}^2 + \left[\frac{ml^2}{2r^2} + V(r)\right] = E$$

Recall that the term in square brackets is $U(r)$. Then

$$m\ddot{r} = -\frac{dU}{dr} = \frac{ml^2}{r^3} - \frac{dV}{dr} \quad \Longrightarrow \quad \boxed{m\ddot{r} = \frac{ml^2}{r^3} + f(r)}$$

For a circular orbit, $\dot{r} = \ddot{r} = 0$ since $r = a$ is constant. Plugging this in yields

$$f(a) = -\frac{ml^2}{a^3}$$

We now expand r about a: $x = r - a$

$$\Longrightarrow \quad m\ddot{x} = ml^2\frac{1}{(x+a)^3} + f(x+a)$$

$$m\ddot{x} = \frac{ml^2}{a^3}\left[\frac{1}{1+x/a}\right]^3 + f(a) + f'(a)x + ...\sigma(x^2)$$

$$m\ddot{x} = \frac{ml^2}{a^3}\left(1 - \frac{3x}{a}\right) + f(a) + f'(a)x + ...$$

Note that $ml^2/a^3 = -f(a)$ and hence

$$m\ddot{x} = \left[f'(a) + \frac{3}{a}f(a)\right]x + ...$$

$$\boxed{m\ddot{x} + \left[-\frac{3}{a}f(a) - f'(a)\right]x = 0}$$

If the term in square brackets $[\cdot]$ is positive then the particle will oscillate about $r = a$ and the orbit is stable. If $[\cdot]$ is negative then $r \sim e^t$ and the radius will blow up.

Criteria for Stability: $f(a) + (a/3)f'(a) < 0$

Example: $f(r) = -cr^m \quad (c > 0)$

$$\Longrightarrow \quad -ca^m + \frac{a}{3}(-cma^{m-1}) \;=\; -ca^m\left[1 + \frac{m}{3}\right] < 0$$

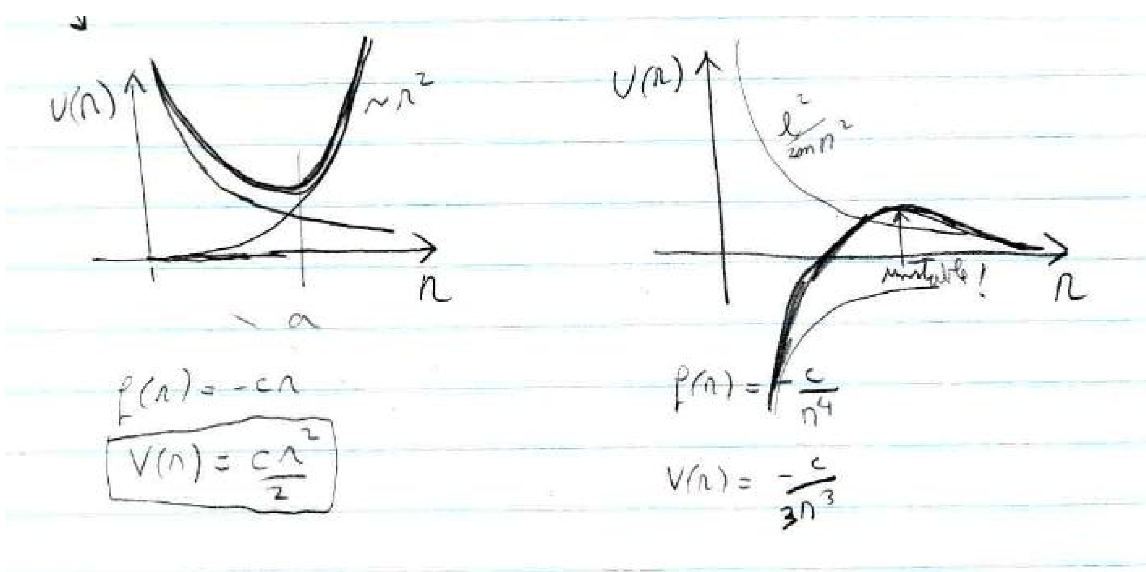
This is stable when

$$1 + \frac{m}{3} > 0 \quad \Longrightarrow \quad \boxed{m > -3}$$

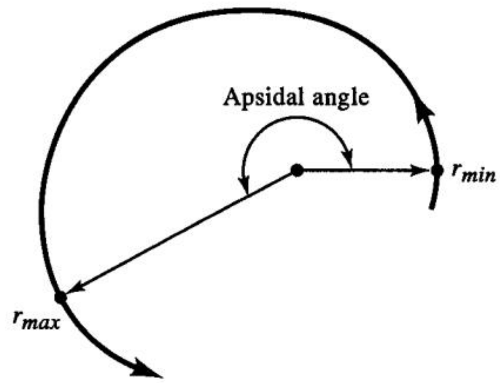
One may show that $f(r) = -c/r^3$ is *unstable*.

Stable Circular Orbits:

$f(r) = -cr \quad$ (2d harmonic oscillator)
$f(r) = -c/r^2 \quad$ (gravity, ...)



Apsides and Apisodal Angles for Nearly Circular Orbits



Apisidal Angle: Polar angle between r_{min} and r_{max}. We give it the symbol ψ. This is a very important quantity for characterizing effects of other planets (e.g. Jupiter) on a planet's orbit. Note that $\psi = \pi$ for an ellipse.

When the orbit is stable, we showed that r will oscillate about $r = a$ with period

$$\tau_r = 2\pi\sqrt{\frac{m}{[-\frac{3}{a}f(a) - f'(a)]}}$$

The apisidal angle ψ is then the value that θ changes from $t = 0$ to $t = \tau_r/2$ (half a period). Now since

$$\dot{\theta} = \frac{l}{r^2} \approx \frac{l}{a^2} = \frac{\sqrt{-\frac{a^3 f(a)}{m}}}{a^2} = \sqrt{-\frac{a^3 f(a)}{ma^4}} = \sqrt{-\frac{f(a)}{ma}}$$

$$\implies \quad \frac{d\theta}{dt} = \sqrt{-\frac{f(a)}{ma}} \quad \implies \quad \int_0^{\psi} d\theta = \int_0^{\tau_r/2} dt\sqrt{-\frac{f(a)}{ma}}$$

$$\implies \quad \psi = \frac{\tau_r}{2}\sqrt{-\frac{f(a)}{ma}} \;=\; \pi\sqrt{\frac{f(a)}{3f(a) + af'(a)}} \;=\; \pi\left[3 + \frac{af'(a)}{f(a)}\right]^{-1/2}$$

For example, when $f(r) = -cr^n$, we get

$$\psi = \pi\left[3 + \frac{a(-cn)a^{n-1}}{-ca^n}\right] \;=\; \frac{\pi}{\sqrt{3+n}}$$

This is independent of the size of the orbit! For $n = -2$ (inverse square law), $\psi = \pi$ as expected. For $f(r) = -cr$ (harmonic 2d), $\psi = \pi/2 \implies 2\pi/\psi = 4 =$ integer and hence the orbit is repetitive.

On the other hand, for $n = 2$, we find that $\psi = \pi/\sqrt{5}$ and hence the orbit never repeats itself.

If the law of forces departs slightly from the inverse square law ($n = 2$), then the apsides either advances or regresses steadily, depending on whether $\tau \gtrsim \pi$ or $\tau \lesssim \pi$.

This is easy to measure! Consider the effect of other planets on planet Mercury. Mercury is subject to a force

$$f(r) = -\frac{k}{r^2} + \epsilon r \qquad (\epsilon > 0)$$

The term ϵr is the repulsion due to a "uniform ring" of mass (other planets, mainly Jupiter).

$$\psi = \pi \left[3 + \frac{a(-\frac{a^2}{k})(\frac{2k}{a^3} + \epsilon)}{(-\frac{a^2}{k})(-\frac{k}{a^2} + \epsilon a)}\right]^{-1/2} = \pi \left[3 + \frac{-2 - \frac{\epsilon a^3}{k}}{1 - \frac{\epsilon a^3}{k}}\right]^{-1/2}$$

$$= \pi \left[\frac{1 - \frac{4\epsilon a^3}{k}}{1 - \frac{\epsilon a^3}{k}}\right]^{-1/2} = \pi \left(1 - \frac{\epsilon a^3}{k}\right)^{1/2} \left(1 - \frac{4\epsilon a^3}{k}\right)^{-1/2}$$

$$= \pi \left(1 - \frac{1}{2}\frac{\epsilon a^3}{k} + ...\right)\left(1 + \frac{2\epsilon a^3}{k} + ...\right) = \pi \left(1 + \frac{3}{2}\frac{\epsilon a^3}{k}\right) > \pi$$

Hence the perihelion advances!

In 1877, people succeeded in calculating the effect of all known planets on one another's orbits. The apisodal angles were found to advance or regress depending on the planet. All observations of the planets agreed nicely with the theory........except mercury.

The perihelion of mercury advances by $575''$ every century. ($1'' = 1^\circ/3600$). The theory only predicted $534''$.

So is planet Vulcan within Mercury's orbit? Actually no! It's due to the effects of general relativity, as shown by Einstein. This was an important confirmation of general relativity.