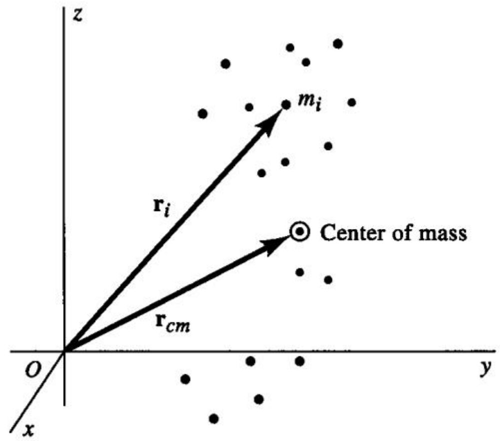
Lecture 22: Dynamics of Systems of Particles



We have n particles, each of mass m_i. The center of mass depends on

$$\vec{R}_{cm} = \frac{1}{M}\sum_i m_i\vec{r}_i$$

where $M = \sum_i m_i$. Note that

$$x_{cm} = \frac{1}{M}\sum_i m_i x_i \quad , \quad y_{cm} = \frac{1}{M}\sum_i m_i y_i \quad , \quad z_{cm} = \frac{1}{M}\sum_i m_i z_i \quad ,$$

Linear Momentum

$$\vec{p} = \sum_i \vec{p}_i = \sum_i m_i\vec{v}_i$$

Since $\dot{\vec{R}_{cm}} = \sum_i m_i\vec{v}_i$ we have that $\vec{p} = M\dot{\vec{R}}_{cm}$ or that

$$\boxed{\vec{p} = M\vec{v}_{cm}}$$

$\Longrightarrow$ The linear momentum of a system of particles is equal to the velocity of the center of mass multiplied by the total mass.

Newton's second law for each particle i:

$$\vec{F}_i + \sum_j \vec{F}_{ij} = m_i \ddot{\vec{r}}_i = \dot{\vec{p}}_i$$

where $\vec{F}_i$ is the external force acting on particle i, and $\vec{F}_{ij}$ is the inter-particle interactions. If we sum over all the i's:

$$\sum_i \vec{F}_i + \sum_{i,j} \vec{F}_{ij} = \sum_i m_i \ddot{\vec{r}}_i = \sum_i \dot{\vec{p}}_i = \dot{\vec{p}}_{cm}$$

By Newton's Third law, $\vec{F}_{ij} = -\vec{F}_{ji}$ and so

$$\sum_{i,j} \vec{F}_{ij} = 0$$

Hence we have

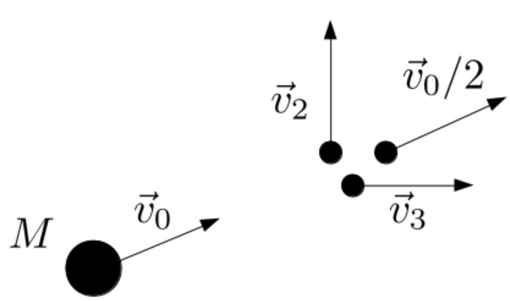
$$\boxed{\sum_i \vec{F}_i = M\vec{A}_{cm} = \dot{\vec{p}}_{cm}}$$

In the case that no external forces are acting, $d\vec{p}_{cm}/dt = 0$. It follows that

$$\boxed{\text{Newton's third law } \textit{implies} \text{ conservation of linear momentum for an isolated system}}$$

(since $\vec{p}_{cm}$ is constant and independent of time)

Example: At some point in its trajectory a ballistic missile of mass M breaks into three fragments- each with mass $M/3$. One fragment continues with initial velocity equal to $\vec{v}_0/2$, where $\vec{v}_0$ is the missile trajectory just before break up. The other two pieces go off at right angles to each other with equal speeds. Find the initial speed of each of the fragments in terms of v_0.



The equations of linear momentum at breakup are

$$M\vec{v}_{cm} = M\vec{v}_0 = \frac{M}{3}\vec{v} + \frac{M}{3}\vec{v}_2 + \frac{M}{3}\vec{v}_3$$

$$= \frac{M}{6}\vec{v}_0 + \frac{M}{3}(\vec{v}_1 + \vec{v}_3))$$

Hence

$$\left(3 - \frac{1}{2}\right)\vec{v}_0 = (\vec{v}_2 + \vec{v}_3) \quad \Longrightarrow \quad \left|\frac{5}{2}\vec{v}_0\right|^2 \;=\; |(\vec{v}_2 + \vec{v}_3)|^2$$

$$\Longrightarrow \quad \frac{25}{4}v_0^2 \;=\; v_2^2 + v_3^2 + 2\vec{v}_2 \cdot \vec{v}_3 \;=\; 2v_2^2$$

It follows that

$$\boxed{v_2 = \frac{5}{2\sqrt{2}}v_0 \approx 1.77v_0}$$

Angular Momentum

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

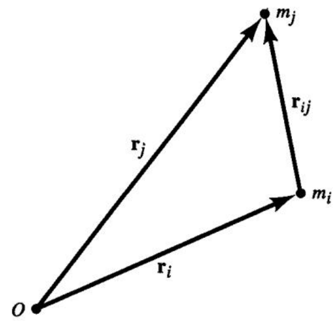
$$\Longrightarrow \quad \frac{d\vec{L}}{dt} = \sum_i \vec{v}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \dot{\vec{p}}_i$$

But $\vec{v}_i \times \vec{p}_i = \vec{v}_i \times m_i\vec{v}_i = 0$ and $\dot{\vec{p}}_i = \vec{F}_i + \sum_j \vec{F}_{ij}$ and hence

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i + \sum_{i,j} \vec{r}_i \times \vec{F}_{ij}$$

Note that

$$\sum_{i,j} \vec{r}_i \times \vec{F}_{ij} = \sum_{i<j}(\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j + \vec{F}_{ji}) = \sum_{i<j}(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}$$

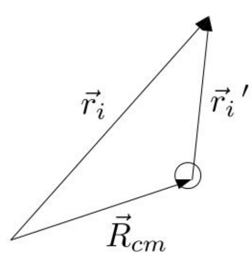


We see from this diagram that if the forces are central between particles, then F_{ij} is parallel to $\vec{r}_i - \vec{r}_j$ and hence $(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} = 0$

$$\implies \boxed{\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i = \vec{N}} \qquad \text{(Torque)}$$

Again, if the system is isolated then $d\vec{L}/dt = 0$ and $\vec{L}$ is constant.

Expressing $\vec{L}$ in terms of the c_m:



$$\begin{cases} \vec{r}_i = \vec{R}_{cm} + \vec{r}\,' \\ \vec{v}_i = \vec{v}_{cm} + \vec{v}_i\,' \end{cases}$$

$$\vec{L} = \sum_i (\vec{R}_{cm} + \vec{r}_i\,') \times \left(\frac{m_i}{M}\vec{p}_{cm} + \vec{p}_i'\right)$$

$$= \left(\vec{R}_{cm} \times \frac{\sum_i m_i}{M}\vec{p}_{cm}\right) + \left(\vec{R}_{cm} \times \sum_i \vec{p}_i\,'\right) + \left(\left(\frac{\sum_i \vec{r}_i\,'}{M}\right) \times \vec{p}_{cm}\right) + \left(\sum_i \vec{r}_i\,' \times \vec{p}_i\,'\right)$$

We analyze each term of the expression. In the second term,

$$\begin{aligned}\sum_i \vec{p}_i\,' &= \sum_i m_i \vec{v}_i\,' \\ &= \sum_i (\vec{v}_i - \vec{v}_{cm}) \\ &= \sum_i m_i \vec{v}_i - m\vec{v}_{cm} \\ &= \vec{p}_{cm} - \vec{p}_{cm} = 0\end{aligned}$$

and hence the second term is equal to zero. In the third term

$$\left(\frac{\sum_i \vec{r}_i\,'}{M}\right) = \vec{R}'_{cm} = 0$$

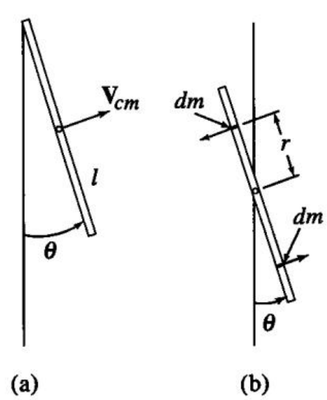
and hence the third term is equal to zero. It follows that

$$\boxed{\vec{L} = (\vec{R}_{cm} \times \vec{p}_{cm}) + \sum_i \vec{r}_i\,' \times \vec{p}_i\,' = \vec{L}_{cm} + \vec{L}_{relcm}\,'}$$

$\vec{R}_{cm} \times \vec{p}_{cm}$ corresponds to the "orbital" motion of the center of mass.

$\sum_i \vec{r}_i\,' \times \vec{p}_i\,'$ corresponds to the "spin part" motion about the center of mass.

Example: A long, thin rod of mass M and length L hangs from one of its ends. Calculate the total $\vec{L}$ of the rod as a function of angular velocity ω.



From (a):

$$L_{cm} = |\vec{R}_{cm} \times \vec{p}_{cm}| = \frac{L}{2} M \frac{L}{2} \omega = \frac{1}{4} M L^2 \omega$$

From (b):

$$dL_{rel} = 2rv\ dm = 2r(r\omega)(\lambda dr) = 2r^2 \omega \lambda dr$$

$$\implies L_{rel} = \int_0^{L/2} 2\omega\lambda r^2 dr = 2\omega\lambda(L^3/24) \ = \ \frac{1}{2} M L^2 \omega = I_{cm}\omega$$

It follows that

$$L_{tot} = L_{cm} + L_{rel} = \left(\frac{1}{12} + \frac{1}{4}\right) M L^2 \omega \ = \ \frac{1}{3} M L^2 \omega$$

Note that $I_{tot} = (1/3)ML^2$.