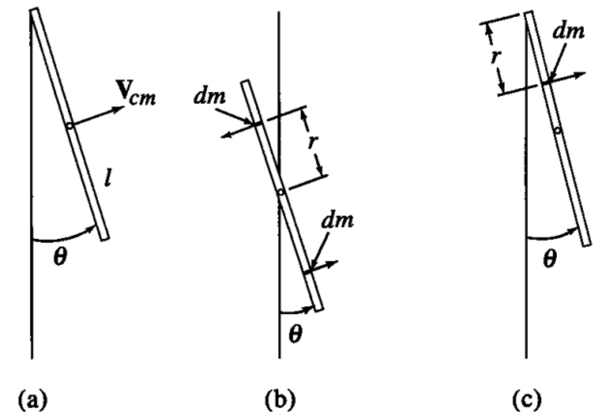
Lecture 23: Dynamics of Systems of Particles Continued

Example: A long, thin rod of mass M and length L hangs from one of its ends. Calculate the total $\vec{L}$ of the rod as a function of angular velocity ω.



From (a):

$$L_{cm} = |\vec{R}_{cm} \times \vec{p}_{cm}| = \frac{L}{2} M \frac{L}{2} \omega = \frac{1}{4} M L^2 \omega$$

From (b):

$$dL_{rel} = 2rv\ dm = 2r(r\omega)(\lambda dr) = 2r^2\omega\lambda dr$$

$$\implies L_{rel} = \int_0^{L/2} 2\omega\lambda r^2 dr = 2\omega\lambda(L^3/24) \;\; = \;\; \frac{1}{2} M L^2 \omega = I_{cm}\omega$$

It follows that

$$L_{tot} = L_{cm} + L_{rel} = \left(\frac{1}{12} + \frac{1}{4}\right) M L^2 \omega \;\; = \;\; \frac{1}{3} M L^2 \omega$$

We can also calculate L_{tot} directly from (c):

$$dL_{tot} = rvdm = r(wr)\lambda dr$$

$$\implies L_{tot} = \int_0^L r^2 dr = w\lambda L^3/3 \;\; = \;\; \frac{1}{3} M L^2 \omega$$

Similar Results for Kinetic Energy:

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \vec{v}_i^2 = \frac{1}{2}\sum_i m_i(\vec{v}_{cm} + \vec{v}_i\,') \cdot (\vec{v}_{cm} + \vec{v}_i\,') \\ &= \frac{1}{2}\left(\sum_i m_i\right) v_{cm}^2 + 2\frac{1}{2}\vec{v}_{cm} \cdot \sum_i m_i \vec{v}_i\,' + \frac{1}{2}\sum_i m_i \vec{v}_i^2 \end{aligned}$$

Since $\sum_i m_i \vec{v}_i\,' = 0$ if follows that

$$\begin{aligned} T &= \frac{1}{2}M v_{cm}^2 + \sum_i \frac{1}{2} m_i \vec{v}_i^2 \\ &= T_{cm} + T_{relcm} \end{aligned}$$

Example: Rod of length L

$$T_{cm} = \frac{1}{2}M\left(\frac{L}{2}\omega\right) = \frac{1}{2}\left(\frac{ML^2}{4}\right)\omega^2$$

$$dT_{rel} = 2\frac{1}{2}dmv^2 = dmr^2\omega^2 = \lambda\omega^2 r^2 dr$$

and hence

$$T_{rel} = \lambda\omega^2 \int_0^{L/2} r^2 dr = \lambda\omega^2 \frac{1}{3}\left(\frac{L}{2}\right)^3 = \frac{1}{2}\lambda\omega^3 \frac{L^3}{12} = \frac{1}{2}\left(\frac{ML^2}{12}\right)\omega^2$$

so

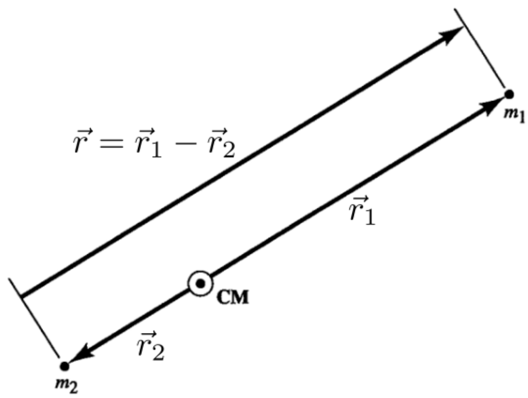
$$T_{tot} = T_{cm} + T_{rel} = \frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2$$

Note that $ML^2/3 = I_{tot}$. We can also find T_{tot} in the following way:

$$dT_{tot} = \frac{1}{2}dm(r\omega)^2 = \frac{1}{2}\lambda\omega^2 r^2 dr$$

$$\implies \quad T_{tot} = \frac{1}{2}\lambda\omega^2 \frac{L^3}{3} = \frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2$$

Motion of Two Interacting Bodies, Reduced Mass



We choose the origin as the CM:

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}$$

$$\Longrightarrow \quad \vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1$$

Describe system by relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$:

$$\vec{r} \;=\; \vec{r}_1 - \vec{r}_2 \;=\; \vec{r}_1\left(1 + \frac{m_1}{m_2}\right)$$

Newton's Second Law for $\vec{r}_1$:

$$m\ddot{\vec{r}}_1 = \vec{F}_{12} = f(r)\frac{\vec{r}}{r}$$

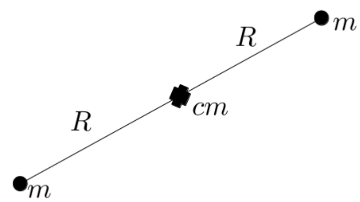
$$\Longrightarrow \quad \frac{m_1\ddot{\vec{r}}}{(1 + \frac{m_1}{m_2})} = f(r)\hat{e}_r \quad \Longrightarrow \quad \left(\frac{m_1 m_2}{m_1 + m_2}\right) = f(r)\hat{e}_r$$

We call $m_1 m_2/(m_1 + m_2)$ the *reduced mass*. The relative motion of both particles occurs about the cm ($\vec{R}_{cm} = \vec{0}$ always) and the relative displacement moves with a reduced mass

$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$$

For example, if the bodies have equal mass then $\mu = m/2$.

Example: Find the velocity required for two objects of mass m to move in a circle of radius R about their cm when they attract according to gravity.



$$|\vec{F}_{12}| = \frac{Gm^2}{(2R)^2} = \frac{1}{4}\frac{Gm^2}{R^2}$$

$$\frac{mv^2}{R} = \frac{1}{4}\frac{Gm^2}{R} \quad \Longrightarrow \quad \boxed{v = \frac{1}{2}\sqrt{\frac{Gm}{R}}}$$

Other method:

$$\mu\ddot{\vec{r}} = \frac{Gm^2}{r^2} \quad \rightarrow \quad \text{since } \ddot{\vec{r}} = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = [a_c - (-a_c)]\hat{e}_r = 2a_c\hat{e}_r$$

$$\Longrightarrow \quad \left(\frac{m}{2}\right)(2a_c) = \frac{Gm^2}{(2R)^2} \quad \text{(same as above)}$$

Connection to Kepler's Third Law

In 6.6 we showed that

$$\tau^2 = 4\pi^2 a^3 \frac{m}{k}$$

if the sun remains stationary. The starting point was the equation

$$m\ddot{\vec{r}}_1 = -\frac{k}{r_1^2} \quad , \quad k = GmM_\odot$$

In actuality, the sun does not remain stationary; it moves, but the coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ the following equation

$$\mu\ddot{\vec{r}} = -\frac{k}{r^2} \quad \Longrightarrow \quad \left|\frac{mM_\odot}{m+M_\odot}\right| = -\frac{gmM_\odot}{r^2} \quad \Longrightarrow \quad m\ddot{\vec{r}} = -\frac{Gm(m+M_\odot)}{r^2}$$

Kepler's third law becomes

$$\tau^2 = a^3 4\pi^2 \frac{m}{Gm(m + M_\odot)}$$

$$\boxed{\tau^2 = a^3 \left(\frac{4\pi^2}{G(m + M_\odot)}\right)}$$

Now the coefficient depends on m and is no longer universal! The dependence, however, is very weak. Take Jupiter for example

$$\frac{\text{coeff corrected}}{\text{coeff without correction}} = \frac{\frac{4\pi^2}{G(m+M_\odot)}}{\frac{4\pi^2}{GM_\odot}} = \frac{M_\odot}{m + M_\odot} \approx 1 - \frac{m}{M_\odot} = 1 - 10^{-3} = 0.999$$

This corresponds to a 0.1% correction at most!