Lecture 24

Binary Stars and Black Holes

Half of the stars in the galaxy in the region near the sun are binary or double.

Example: Sirius, brightest star in the sky

$$\begin{aligned} & M_1 = 2.1 M_\odot \\ & M_2 = 1.05 M_\odot \quad \text{(white dwarf)} \\ \Longrightarrow\ & a = 20 \text{ A.U.} \end{aligned}$$

Period?

$$\tau = a^{3/2} \frac{1}{(m_1 + m_2)^{1/2}} \quad = \quad \frac{(20)^{3/2}}{(3.15)^{1/2})} \text{ years} \quad = \quad 50.4 \text{ years}$$

Another interesting binary system is Cygnus x-1. It is the strangest X-Ray source seen from earth and is formed by a normal star (HDE 226868) and a strange companion. Consider the optical spectroscopic observations:

$$\begin{cases} \tau & = \quad 5.6 \text{ days} \quad = \quad 1.53 \times 10^{-2} \text{ years} \\ a & = \quad 30 \times 10^6 \text{ km} \quad = \quad 0.2 \text{ A.U.} \end{cases}$$

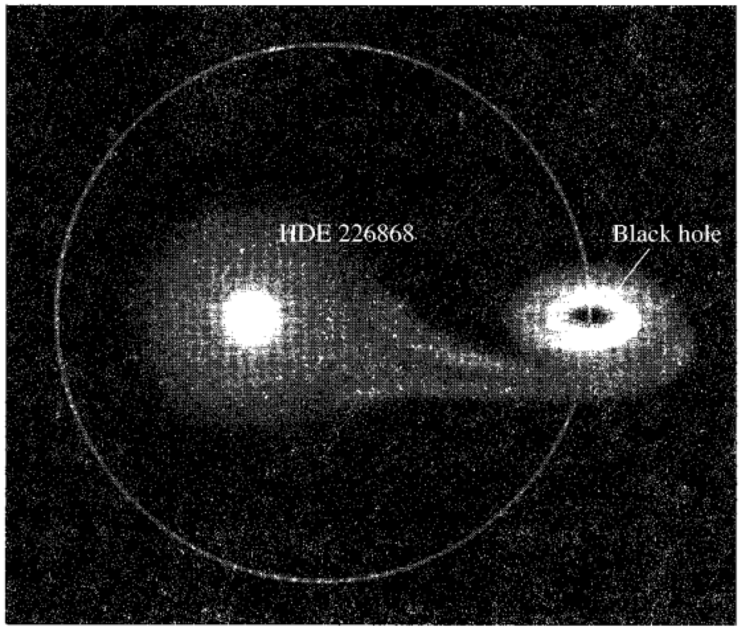
Star $\rightarrow m_1 = 25.5 M_\odot$. What is m_2?

$$(m_1 + m_2) \quad = \quad \frac{a^3}{\tau^2} \quad = \quad \frac{0.2^3}{(1.53 \times 10^{-2})^2} \quad = \quad 34.2$$

$$\Longrightarrow \quad m_2 = 34.2 - 25.5 = 8.7 M_\odot$$

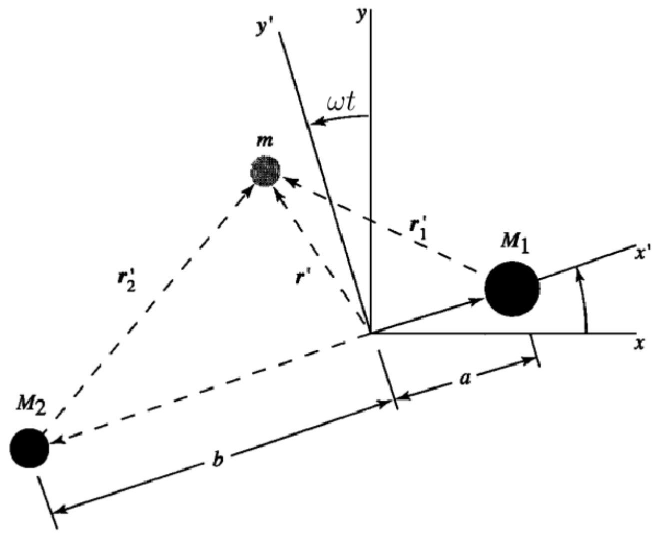
It turns out the Cygnus x-1 is the strangest x-ray source seen from earth. What is the source of the x-rays?

The radius of m_2 is only 26 km! Thats about the distance from campus to the airport! A key astronomical finding is that the radius of the companion is too small to be a normal star. It is believed that this is a black hole.



→ Subject to a funny bet between Stephen Hawking and Kip Thorne. (Kip Thorne won a subscription to Penthouse Magazine).

The Restricted Three Body Problem



Restrictions:

(i): $\begin{cases} m << M_1 \\ m << M_2 \end{cases}$ (m is called the "tertiary")

(ii) All orbits lie in the same plane, and each of the orbits of the two primaries can be approximated by a circle.

We will assume that the motion of M_1, M_2 is *unaffected* by m. However, m orbits around M_1, M_2 and M_1, M_2 orbit about the center of mass with frequency ω. Hence we use the rotating coordinate system shown above.

Coordinates of m: (x', y')

$$\begin{aligned} {r_1}' = |\vec{r}\,' - \vec{a}| &= \sqrt{(x'-a)^2 + y'^2} \quad , \quad \vec{a} = a\hat{x}\,' \\ {r_2}' = |\vec{r}\,' - \vec{b}| &= \sqrt{(x'+b)^2 + y'^2} \quad , \quad \vec{b} = -b\hat{x}\,' \end{aligned}$$

$$\vec{F} = -m\frac{GM_1}{r_1'^2}\frac{\vec{r_1}\,'}{r_1'} - m\frac{GM_2}{r_2'^2}\frac{\vec{r_2}\,'}{r_2'}$$

We write the equations of motion for the rotating reference frame:

$$\vec{F}\,' = m\vec{a}\,' \;=\; \vec{F} - 2m\vec{\omega}\times\vec{v}\,' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}\,')$$

where

$$2\vec{\omega}\times\vec{v} \;=\; 2\omega\hat{k}\,'\times(\dot{x}\,'\hat{\imath}\,' + \dot{y}\,'\hat{\jmath}\,') \;=\; 2\omega(\dot{x}\,'\hat{\jmath} - \dot{y}\,'\hat{\imath}\,')$$

and

$$\begin{aligned}\vec{\omega}\times(\vec{\omega}\times\vec{r}\,') &= \omega^2\hat{k}\,'\times[\hat{k}\,'\times(x'\hat{\imath}\,' + y'\hat{\jmath}\,')] \;=\; \omega^2\hat{k}\,'\times[x'\hat{\jmath}\,' - y'\hat{\imath}\,'] \\ &= \omega^2(-x'\hat{\imath}\,' - y'\hat{\jmath}\,') \;=\; -\,\omega^2(x'\hat{\imath}\,' + y'\hat{\jmath}\,')\end{aligned}$$

The equation of motion becomes

$$\ddot{\vec{r}}\,' \;=\; -\frac{GM_1}{r_1'^3}\vec{r}_1\,' - \frac{GM_2}{r_2'^3}\vec{r}_2\,' - 2\omega\dot{x}\,'\hat{\jmath}\,' + 2\omega\dot{y}\,'\hat{\imath}\,' + \omega^2(x'\hat{\imath}\,' + y'\hat{\jmath}\,')$$

$$\begin{cases}\ddot{x}' = -GM_1\dfrac{(x'-a)}{[(x'-a)^2+y'^2]^{3/2}} - GM_2\dfrac{(x'+b)}{[(x'+b)^2+y'^2]^{3/2}} + \omega^2 x' + 2\omega\dot{y}\,' \\[2ex] \ddot{y}' = -GM_1\dfrac{y'}{[(x'-a)^2+y'^2]^{3/2}} - GM_2\dfrac{y'}{[(x'+b)^2+y'^2]^{3/2}} + \omega^2 y' - 2\omega\dot{x}\,'\end{cases}$$

Effective Potential and the Five Lagrangian Points

The first three terms in the RHS can be written as a potential. LEt $\vec{a} = a\hat{x}$ 'and $\vec{b} = -b\hat{x}\,'$.

$$\boxed{V(\vec{r}\,') = -\frac{GmM_1}{|\vec{r}\,' - \vec{a}|} - \frac{GmM_2}{|\vec{r}\,' - \vec{b}|} - \frac{1}{2}m\omega^2 r'^2} \quad , \quad \text{so that}$$

$$\boxed{m\ddot{\vec{r}} = -\vec{\nabla}V(\vec{r}\,') - 2m\vec{\omega}\times\vec{v}\,'}$$

The first two terms of $V(\vec{r}\,')$ are the gravitational terms and the final term is the centrifugal term. Let's think about the term $-\vec{\nabla}V(\vec{r}\,')$.

For convenience, let's choose units such that:

$$\text{Set}\quad \begin{cases} a+b = 1 \text{ length unit} \\ G(m_1+m_2) = 1 \text{ mass unit} \\ \tau \quad , \quad \text{orbital period of the primaries} = 2\pi \text{ time units} \quad (\text{or } \omega = 1 \text{ inverse time unit}) \\ \text{Measure energy in units of } G(m_1+m_2)/(a+b) \end{cases}$$

We characterize the equations of motion by a single parameter:

$$\boxed{\alpha = \frac{M_1}{(M_1+M_2)}} \in [0, 1/2] \quad (M_2 \leq M_1)$$

$$\implies \quad \text{since } x_{cm} = \frac{M_1 a}{M_1+M_2} - \frac{M_2 b}{M_1+M_2} = 0 \quad \implies \quad \boxed{\frac{a}{b} = \frac{M_2}{M_1}}$$

It follows that

$$\frac{a}{a+b} \quad = \quad \frac{a/b}{1+a/b} \quad = \quad \frac{M_2/M_1}{1+M_2/M_1} \quad = \quad \frac{M_2}{M_1+M_2} \quad = \quad \alpha \quad \left(\text{and } \frac{b}{a+b} = 1-\alpha\right)$$

and hence

$$\frac{V(x',y')}{G(M_1+M_2)/(a+b)}$$

$$= \frac{M_1}{M_1+M_2}\left(\frac{1}{\sqrt{\left(\frac{x'}{a+b}-\frac{a}{a+b}\right)^2+\left(\frac{y'}{a+b}\right)^2}}\right) - \frac{M_2}{M_1+M_2}\left(\frac{1}{\sqrt{\left(\frac{x'}{a+b}+\frac{b}{a+b}\right)^2+\left(\frac{y'}{a+b}\right)^2}}\right)$$

$$-\frac{1}{2}\,\frac{\omega^2(a+b)^2}{G(M_1+M_2)/(a+b)}\left(\frac{r'}{a+b}\right)$$

We measure x', y' in units of $(a+b)$, V in units of...

$$\boxed{V(x',y') = -\frac{(1-\alpha)}{\sqrt{(x'-a)^2+y'^2}} - \frac{\alpha}{\sqrt{(x'+1-\alpha)^2}} - \frac{x'^2+y'^2}{2}}$$

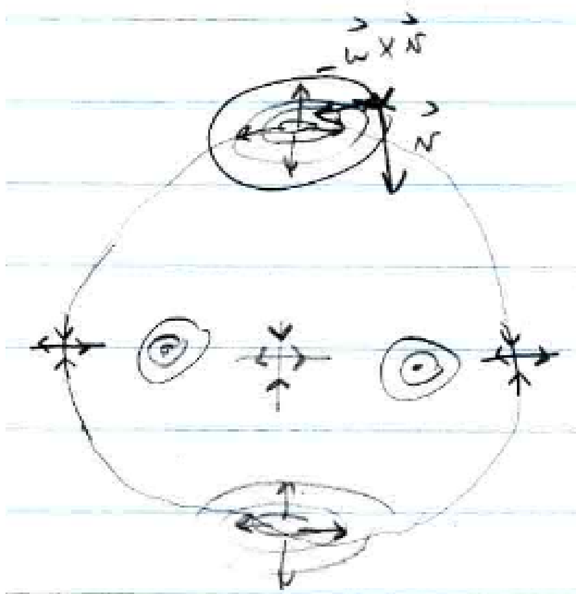
(i) Show plot in Mathematica when $\alpha = 1/2$. $V \to -\infty$ at each primary location. m may orbit each primary and will get dragged along by it just as the moon orbits the earth. Argue that there are some equilibrium points where $\vec{\nabla} = \vec{0}$. Stable or unstable?

(ii) Show contour plot. Show that there are 5 Lagrangian points where $\vec{\nabla}V = 0$.
$\implies$ 3 saddle points along x' $(L_1 - L_3)$
$\implies$ 2 unstable (max) points along y' $(L_4 - L_5)$

(iii) Is it possible for a tertiary to remain locked at any of these five points, synchronously locked to the two primaries as they rotate about the center of mass?

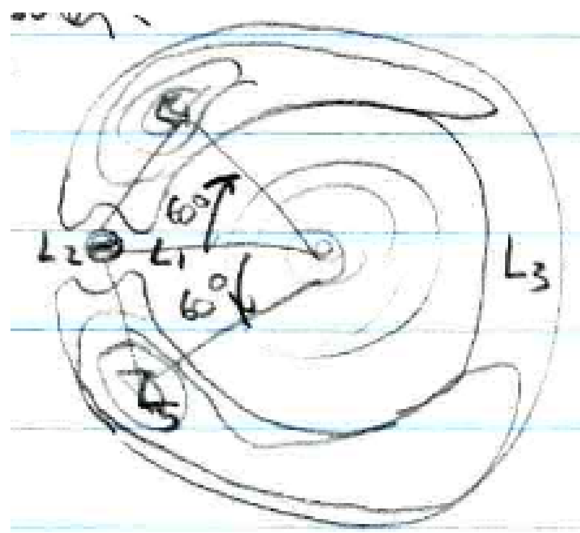
This can't happen for $L_1 - L_3$ (along x') $\implies$ Saddle points! Any perturbation destabilizes the tertiary.

However, V is rather flat around L_4 and L_5. But V is a local maximum, can it be stable? If the object rotates *clockwise*, the Coriolis force stabilizes the orbit!



(iv) Describe case $M_1 >> M_2$, e.g. Sun and Jupiter.

(v) L_4 and L_5 are now forming equilateral triangles.



Show distribution of asteroids in the solar system!