## Lecture 25: Collisions



Without Contact


With Contact

Conservation of Momentum: $m_{1} \vec{v}_{1}+m_{2} \vec{v}_{2}=m_{1} \vec{v}_{1}{ }^{\prime}+m_{2} \vec{v}_{2}{ }^{\prime}$

Conservation of Energy: $\frac{1}{2} m_{1} v_{1}^{2}+\frac{1}{2} m_{2} v_{2}^{2}=\frac{1}{2} m_{1} v_{1}{ }^{\prime 2}+\frac{1}{2} m_{2} v_{2}{ }^{\prime 2}+Q$

Note that $Q>0$ for kinetic energy loss (sound, deformation, etc..) and $Q<0$ for kinetic energy gain (for example if the particles exploded when they touched).

The power of high energy physics is able to probe the fundamental constituents of matter by colliding particles with each other at very high energies. This is what occurs at the ATLAS experiment at the Large Hadron Collider (LHC). The LHC collides protons head on, each proton with energy $7 \mathrm{TeV}=7 \times 10^{12} \mathrm{eV}=1.12 \times 10^{-6} \mathrm{~J}$. The velocity of each proton is $0.99999999 c=\left(1-10^{-8}\right) c$.

## Collisions in 1D

Before


After


$$
m_{1} \dot{x}_{1}+m_{2} \dot{x}_{2}=m_{1} \dot{x}_{1}^{\prime}+m_{2} \dot{x}_{2}^{\prime}
$$

$$
\frac{1}{2} m_{1} \dot{x}_{1}^{2}+\frac{1}{2} m_{2} \dot{x}_{2}^{2}=\frac{1}{2} m_{1} \dot{x}_{1}^{\prime 2}+\frac{1}{2} m_{2} \dot{x}_{2}^{\prime 2}+Q
$$

Instead of introducing Q , it is more convenient to define the coefficient of restitution $\epsilon$ :

$$
\epsilon=\left|\frac{\dot{x}_{2}^{\prime}-\dot{x}_{1}^{\prime}}{\dot{x}_{2}-\dot{x}_{1}}\right|=\left|\frac{v}{v^{\prime}}\right|
$$

$v$ and $v^{\prime}$ are the "before" and "after" velocities of $m_{2}$ for an observer siting at $m_{1}$.

Totally Inelastic Collision: Objects stick together $\Longrightarrow \dot{x}_{1}{ }^{\prime}=\dot{x}_{2}{ }^{\prime} \Longrightarrow \epsilon=0$.

Elastic Collision: May show that $v^{\prime}=-v \Longrightarrow \epsilon=|-1| \Longrightarrow \epsilon=1$. Also, assume $\dot{x}_{2}=0$

$$
\Longrightarrow \quad \dot{x}_{1}^{\prime}=\left(\frac{m_{1}-m_{2}}{m_{1}+m_{2}}\right) \dot{x}_{1} \quad ; \quad \dot{x}_{2}=\left(\frac{2 m_{1}}{m_{1}+m_{2}}\right) \dot{x}_{1}
$$

In the first problem of the homework you are required to show that $Q=\frac{1}{2} \mu v^{2}\left(1-\epsilon^{2}\right)$.
(Note that $Q<0$ is described by $\epsilon>1$ )

## Impulse

$$
\begin{gathered}
\frac{\vec{F}_{21}}{d t}=\vec{F} \Longrightarrow \int_{i}^{f} d \vec{p}_{i}=\int_{i}^{f} \vec{F} d t
\end{gathered}
$$

The first integral is equal to $\vec{p}_{1}^{(f)}-\vec{p}_{1}^{(i)}$ and the second one is equal to $\vec{I}$. Hence

$$
\begin{aligned}
& \vec{p}_{1}^{(f)}-\vec{p}_{1}^{(i)}=\vec{I} \\
& \Longrightarrow \Delta \vec{p}_{1}=\vec{I}
\end{aligned}
$$

## Collisions in 2D



Laboratory Reference frame

$$
\begin{gathered}
\vec{p}_{1}=\vec{p}_{1}^{\prime}+\vec{p}_{2}^{\prime} \\
\left\{\begin{array}{l}
m_{1} \vec{v}_{1}=m_{1} \vec{v}_{1}^{\prime}+m_{2} \vec{v}_{2}^{\prime} \\
\frac{p_{1}^{2}}{2 m_{1}}=\frac{p_{1}^{2}}{2 m_{1}}+\frac{p_{2}^{2}}{2 m_{2}}+Q
\end{array}\right.
\end{gathered}
$$



## Center of Mass Frame

(Angles are the same because of conservation of momentum $\Longrightarrow \tan \theta=\tan \theta^{\prime}$ ). Easy to do theory!

$$
\begin{aligned}
& \left\{\begin{array}{l}
\vec{p}_{1}^{\prime} \sin \theta=\vec{p}_{2}^{\prime} \sin \theta^{\prime} \\
\vec{p}_{1}^{\prime} \\
\\
\\
\\
\\
\Longrightarrow \quad \tan \theta=\vec{p}_{2}^{\prime} \\
\end{array} \cos \theta^{\prime}\right.
\end{aligned}
$$

$\underline{m_{1}=m_{2} \text { case: }}$

$$
\begin{gathered}
\left\{\begin{array}{r}
p_{1}{ }^{2}=p_{1}^{\prime 2}+p_{2}^{\prime 2}=2 m Q \\
\vec{p}_{1}=\vec{p}_{1}^{\prime}+\vec{p}_{2}^{\prime} \Longrightarrow p_{1}{ }^{\prime}=p_{1}^{\prime 2}+{p_{2}^{\prime}}^{\prime 2}+2 \vec{p}_{1}^{\prime} \cdot \vec{p}_{2}^{\prime} \quad(* *) \\
\Longrightarrow(*)-(* *)=0=2 m Q-2 \vec{p}_{1}^{\prime} \cdot \vec{p}_{2}^{\prime} \\
Q=\frac{\vec{p}_{1}^{\prime} \cdot \vec{p}_{2}^{\prime}}{m}
\end{array}\right. \\
\qquad
\end{gathered}
$$

For an elastic collision, $Q=0$ and $\vec{p}_{1}{ }^{\prime} \cdot \vec{p}_{2}{ }^{\prime}=0$ Hence if $\psi$ is the opening angle we have

$$
\psi=\phi_{1}+\phi_{2}=\frac{\pi}{2}
$$



General Case, $m_{1} \neq m_{2}$ : How does $\theta$ depend on $\phi_{1}, \phi_{2}$ ? Let a bar above a vector imply that it is in the CM frame.

$$
\begin{aligned}
& \text { In the CM Frame }\left\{\begin{array}{l}
\overrightarrow{p_{1}}+\overrightarrow{p_{2}}=\overrightarrow{0} \\
\overrightarrow{\vec{p}_{1}^{\prime}}+\overrightarrow{\vec{p}_{2}^{\prime}}=\overrightarrow{0} \\
\frac{\bar{p}_{1}^{2}}{2 m_{1}}+\frac{\bar{p}_{2}^{2}}{2 m_{2}}=\frac{\overline{\bar{p}_{1}^{\prime}}}{2 m_{1}}+\frac{\overline{p_{1}}{ }^{\prime 2}}{2 m_{1}}+Q
\end{array}\right. \\
& \Longrightarrow \frac{1}{2}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right){\overline{p_{1}}}^{2}=\frac{1}{2}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right){\overline{p_{1}}}^{\prime 2}+Q
\end{aligned}
$$

Letting $1 / \mu=1 / m_{1}+1 / m_{2}$ we have

$$
\frac{\bar{p}_{1}^{2}}{2 \mu}=\frac{{\overline{p_{1}}}^{\prime 2}}{2 \mu}+Q
$$

The $\vec{v}_{c m}$ (W.R.T lab) is

$$
\vec{v}_{c m}=\frac{m_{1}}{m_{1}+m_{2}} \vec{v}_{1} \quad\left(\vec{v}_{2}=0 \quad \text { by assumption }\right)
$$

Hence

$$
\overrightarrow{v_{1}}=\vec{v}_{1}-\vec{v}_{c m}=\left(1-\frac{m_{1}}{m_{1}+m_{2}}\right) \vec{v}_{1}=\frac{m_{2}}{m_{1}+m_{2}} \vec{v}_{1} ; \quad \overrightarrow{v_{1}}={\overrightarrow{v_{1}}}^{\prime}-\vec{v}_{c m}
$$



$$
\left\{\begin{array}{l}
v_{1}^{\prime} \sin \phi_{1}={\overline{v_{1}}}^{\prime} \sin \theta \\
v_{1}^{\prime} \cos \phi_{1}={\overline{v_{1}}}^{\prime} \cos \theta+v_{c m}
\end{array}\right.
$$

We divide,

$$
\tan \phi_{1}=\frac{\sin \theta}{\left(\frac{v_{c m}}{\bar{v}_{1^{\prime}}}\right)+\cos \theta}=\frac{\sin \theta}{\gamma+\cos \theta}
$$

where

$$
\gamma=\frac{m_{1}}{m_{1}+m_{2}} \frac{v_{1}}{\overline{v_{1}^{\prime}}}=\frac{m_{1}}{m_{1}+m_{2}}\left(\frac{m_{1}+m_{2}}{m_{2}} \overline{v_{1}}\right) \frac{1}{\overline{v_{1}^{\prime}}}=\frac{m_{1}}{m_{2}} \frac{\overline{v_{1}}}{\overline{v_{1}^{\prime}}}
$$

$\overline{v_{1}} /{\overline{v_{1}}}^{\prime}$ can be found from energy:

$$
\frac{{\overline{p_{1}}}^{2}}{2 \mu}+\frac{{\overline{p_{1}}}^{\prime 2}}{2 \mu}+Q
$$

## Elastic

$$
\begin{array}{r}
Q=0 \Longrightarrow \bar{p}_{1}={\overline{p_{1}}}^{\prime} \Longrightarrow v_{1} / \bar{v}_{1}{ }^{\prime}=1, \text { so } \\
\\
\tan \phi_{1}=\frac{\sin \theta}{\frac{m_{1}}{m_{2}}+\cos \theta}
\end{array}
$$

we do a similar calculation for $\phi_{2}$ when $m_{1} \neq m_{2}$ to get

$$
\begin{gathered}
\not \tan \phi_{2}=\frac{\sin \theta}{1-\cos \theta} \\
\Longrightarrow \tan \phi_{2}=\cot (\theta / 2) \\
\Longrightarrow \phi_{2}=\left(\frac{\pi}{2}-\frac{\theta}{2}\right)
\end{gathered}
$$

## Two Cases:

$$
\begin{aligned}
& m_{2} \gg m_{1} \Longrightarrow \tan \phi_{1} \approx \tan \theta \Longrightarrow \phi_{1} \approx \theta \\
& m_{2}=m_{1} \Longrightarrow \tan \phi_{1}=\frac{\sin \theta}{1+\cos \theta}=\frac{2 \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)}{2 \cos ^{2}\left(\frac{\theta}{2}\right)}=\tan \left(\frac{\theta}{2}\right) \Longrightarrow \phi_{1}=\frac{\theta}{2}
\end{aligned}
$$

Earlier, we showed (that even when $m_{2} \neq m_{1}$ ) that $\phi_{2}=\frac{1}{2}(\pi-\theta)$. The angle $\phi-\theta$ corresponds to the angle of deflection of $m_{2}$ in the CM frame.

When $m_{1}=m_{2}$ it follows that $\phi_{1}+\phi_{2}=\pi / 2$ as proved eariler!

In the general case

$$
\gamma=\frac{m_{1}}{m_{2}}\left[1-\frac{Q}{T}\left(1+\frac{m_{1}}{m_{2}}\right)\right]^{-1 / 2}
$$

where $T$ is the kinetic energy of the incident particle in the lab frame: $T=\overline{p_{1}} / 2 m_{1}$

## Motion of a Body with Variable Mass (Rocket Motion)

Consider an object that sticks to another as it moves. What is the change in momentum?

## Before



$$
\vec{p}_{i}=m \vec{v}+(\Delta m) \vec{u}
$$

After


$$
\begin{aligned}
\Delta \vec{p}=\vec{p}_{f}-\vec{p}_{i} & =(m+\Delta m)(\vec{v}+\Delta \vec{v})-m \vec{v}-\Delta m \vec{u} \\
& =m \vec{v}+m \Delta \vec{v}+\Delta m \vec{v}+\Delta m \Delta \vec{v}-m \vec{v}-\Delta m \vec{u} \\
& =m \Delta \vec{v}+(\Delta m \vec{v}+\Delta m \Delta \vec{v}-\Delta m \vec{u}) \\
& =m \Delta \vec{v}-\Delta m(\vec{u}-\vec{v})
\end{aligned}
$$

We let $\vec{V}=\vec{u}-\vec{v}$ be the velocity of $\Delta m$ relative to $m$. Hence

$$
\begin{aligned}
& \Delta \vec{p}=(m+\Delta m) \Delta \vec{v}-\Delta m \vec{V} \\
& \frac{\Delta \vec{p}}{\Delta t}=(m+\Delta m) \frac{\Delta \vec{v}}{\Delta t}-\frac{\Delta m}{\Delta t} \vec{V} \\
& \Longrightarrow \frac{d \vec{p}}{d t}=m \frac{d \vec{v}}{d t}-\frac{d m}{d t} \vec{V}=\vec{F}_{e x t}
\end{aligned}
$$

This is the equation of motion for an object with variable mass. Keep in mind that $\vec{V}=\vec{u}-\vec{v}$. Consider the following particular cases.

Case 1: Drop of rain falling through a mist: $\vec{u}=0$ (mist is at rest), so $\vec{V}=-\vec{v}$ :

$$
\vec{F}_{e x t}=m \frac{d \vec{v}}{d t}-\frac{d m}{d t}(-\vec{v})=\frac{d}{d t}(m \vec{v})
$$

Note that this equation is not general! It only applies when $\vec{u}=0$ !

Case 2: Rocket in space, $\vec{V}=$ exhaust velocity, constant, $d m / d t<0$


$$
m \frac{d \vec{v}}{d t}=\vec{V} \frac{d m}{d t}=-\left|\frac{d m}{d t}\right| \vec{V}
$$

Supposing $\vec{V}=-V \hat{i}$ we have

$$
\begin{gathered}
\int_{v_{0}}^{v_{f}} d v=-V \int_{m_{0}}^{m} \frac{d m}{m} \Longrightarrow v_{f}-v_{0}=-V \ln \left(\frac{m}{m_{0}}\right) \\
\Longrightarrow v_{f}=v_{0}+V \ln \left(\frac{m_{0}}{m}\right)
\end{gathered}
$$

Example: What is the payload ratio for putting a rocket into a LEO ( $v \approx 8 \mathrm{~km} / \mathrm{s}$ )
Note that $F_{\text {ext }}=-m g \hat{j}$ is the external force acting on the rocket and $\vec{V}=-V \hat{j}$ (the fuels velocity is downwards). Hence

$$
\begin{gathered}
m \frac{d v}{d t}+V \frac{d m}{d t}=-m g \quad \Longrightarrow \quad \frac{d v}{v}+\frac{d m}{m}=-\frac{g}{V} d t \\
\frac{d v}{v}=-\frac{d m}{m}-\frac{g}{V} d t>0 \quad \text { "Lift Off" }
\end{gathered}
$$

This occurs only when

$$
\frac{|d m|}{m}>\frac{g}{V} d t \Longrightarrow\left|\frac{d m}{d t}\right|>\frac{m_{0}}{\tau_{s}}
$$

where $\tau_{s}=V / g$ is the specific impulse. Typically $V \approx 3000 \mathrm{~m} / \mathrm{s} \Longrightarrow \tau_{s}=300 \mathrm{~s}$. At each 1 s, the rocket has to eject $\approx 1 / 3000$ of its total mass for lift off. This is why the shuttle starts off so slowly! Now

$$
\frac{1}{V} \int_{0}^{v_{f}}=-\int_{m_{0}}^{m_{f}} \frac{d m}{m}-\frac{1}{\tau_{s}} \int_{0}^{\tau_{B}} d t
$$

where we call $\tau_{B}$ the "burn time." The smaller this is, the better.

$$
\frac{v_{f}}{V}=-\ln \left(\frac{m_{f}}{m_{0}}\right)-\frac{\tau_{B}}{\tau_{S}} \quad \Longrightarrow \quad\left(\frac{m_{f}}{m_{0}}\right)=e^{-\left(\frac{v_{f}}{V}+\frac{\tau_{B}}{\tau_{S}}\right)}
$$

Substituting in $v_{f}=8 \mathrm{~km} / \mathrm{s}, V=310 \mathrm{~m} / \mathrm{s}, \tau_{B}=600 \mathrm{~s}$, and $\tau_{s}=300 \mathrm{~s}$, we find that

$$
\begin{aligned}
& \left(\frac{m_{f}}{m_{0}}\right)=e^{-4.67}=0.0094 \\
\Longrightarrow & \frac{m_{0}}{m_{f}}=\frac{m_{\text {fuel }}}{m_{\text {rocket }}+m_{\text {payload }}}+1=106 \\
\Longrightarrow & \frac{m_{\text {fuel }}}{m_{\text {rocket }}+m_{\text {payload }}}=105
\end{aligned}
$$

In other words, it takes 105 kg of fuel per 1 kg of payload!

