Lecture 26: The Principle of Least Action (Hamilton's Principle)

Where does $\vec{F} = m\vec{a}$ or $-\vec{\nabla}V = m\ddot{\vec{r}}$ come from? Is there a more fundamental reason as to why these equations hold?

Define the Lagrangian function as

$$\boxed{\mathcal{L} = T - V}$$

For example, in a conservative system in 1D we would have

$$\mathcal{L}(y, \dot{y}) = \frac{1}{2}m\dot{y}^2 - V(y)$$

Suppose that a particle starts at $y_1 = y(t_1)$ and ends its trajectory at $y_2 = y(t_2)$. The path taken is a particular $(y(t), \dot{y}(t))$ trajectory; the action of this trajectory is given by

$$J = \int_{t_1}^{t_2} \mathcal{L} dt$$

Hamilton's principle states that out of all the infinite family of motions $(y(t), \dot{y}(t))$, the actual motion that takes place is the one for which the action is an extrema:

$$\delta J = \delta \int_{t_1}^{t_2} \mathcal{L} dt = 0$$

i.e., any variation on top of this motion $(y(t) + n(t), \dot{y} + \dot{n})$ with endpoints fixed $(n(t_1) = n(t_2) = 0)$ vanishes in 1st order. The usual situation is that J has a global minimum at the actual trajectory.

Functional Derivative:

$$J = J[y(t)] \quad , \quad \frac{\delta J}{\delta y} = 0$$

Example: Particle in Free Fall

$$\mathcal{L} = \frac{1}{2}m\dot{y}^2 - mgy$$

$$\begin{aligned}
\delta J = \delta\int_{t_1}^{t_2}\mathcal{L}dt &= \delta\int_{t_1}^{t_2}\left[\frac{m}{2}\dot{y}^2 - mgy\right]dt \\
&= \int_{t_1}^{t_2}\left[\frac{m}{2}2\dot{y}\delta y - mg\delta y\right]dt
\end{aligned}$$

Note that $\delta\dot{y} = \frac{d}{dt}(\delta y)$. We integrate the first term by parts $\int udv = uv - \int vdu$. Let $u = m\dot{y}$ and $\frac{d}{dt}(\delta y) = dv$. Then we have

$$\int_{t_1}^{t_2} m\dot{y}\frac{d}{dt}(\delta y)dt \;=\; m\dot{y}(\delta y)\Big|_{t_1}^{t_2} - \int(\delta y)\frac{d}{dt}(m\dot{y})dt$$

Note that $m\dot{y}(\delta y)\Big|_{t_1}^{t_2} = 0$ since $\delta y(t_i) = 0$. Hence

$$\int_{t_1}^{t_2} m\dot{y}\frac{d}{dt}(\delta y)dt \;=\; -\int(\delta y)\frac{d}{dt}(m\dot{y})dt \;=\; -\int m\ddot{y}(\delta y)dt$$

$$\implies \quad \delta J = -\int_{t_1}^{t_2}[m\ddot{y} + mg](\delta y)dt$$

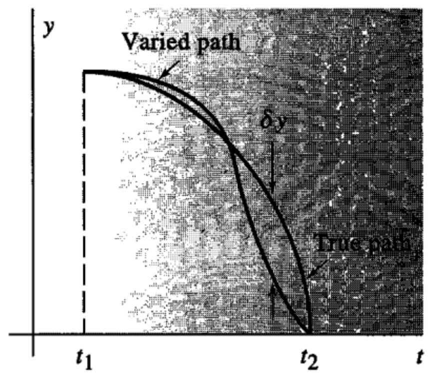
This $= 0$ for any arbitrary (δy) if and only if

$$\boxed{m\ddot{y} = -mg} \quad \text{Newton's 2nd law!}$$

We proved that $\delta J = 0$ for the actual trajectory. Is J a minimum or a maximum?

Actual trajectory: $\boxed{y(t) = -\frac{1}{2}gt^2}$ $\quad (y(t=0)=0, y(t=t_2) = -\frac{1}{2}g{t_2}^2)$

Assume $y(\alpha, t) = y(0,t) + \alpha n(t)$ where $y(0,t)$ is the actual solution, α is a bookkeeping constant, and $n(t)$ is some arbitrary function.



$$J(\alpha) = \int_{t_1}^{t_2} dt \mathcal{L}[y(\alpha,t), \dot{y}(\alpha,t)]$$

$$\begin{cases} \dot{y}(\alpha,t) = \dot{y}(0,t) + \alpha\dot{n}(t) = (-gt + \alpha\dot{n}) \\ T = \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m[-gt+\alpha\dot{n}]^2 \\ V = mgy = mg\left[-\frac{1}{2}gt^2 + \alpha n\right] \end{cases}$$

$$J(\alpha) = \int_{t_1}^{t_2} dt \left(\frac{1}{2}m\left[g^2t^2 - 2gt\alpha\dot{n} + \alpha^2\dot{n}^2\right] - mg[-\frac{1}{2}gt^2 + \alpha n]\right)$$

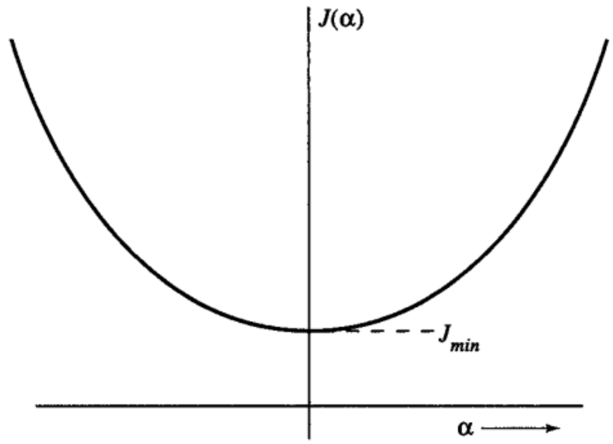
$$= \int_{t_1}^{t_2} dt \left(mg^2t^2 - mg\alpha(t\dot{n} + n) + \frac{1}{2}m\alpha^2\dot{n}^2\right)$$

$$= \int_{t_1}^{t_2} dt \left(mg^2t^2 - mg\alpha(-n + n) + \frac{1}{2}m\alpha^2\dot{n}^2\right)$$

$$= \frac{mg^2}{3}({t_2}^3 - {t_1}^3) + \frac{1}{2}m\alpha^2 \int_{t_1}^{t_2} dt\dot{n}^2 \;\; = \;\; J_0 + J_1\alpha^2$$

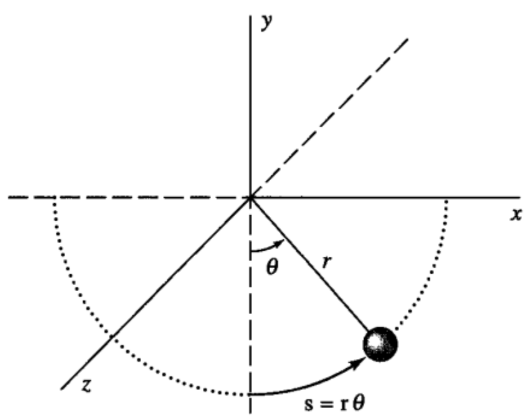
Hence $J(\alpha = 0)$ is a local minimum in the space of all functions $n(t)$!

$$\frac{\partial J(\alpha)}{\partial \alpha}\bigg|_{\alpha=0} = 0$$



Generalized Coordinates

Consider a pendulum in the xy plane. How many degrees of freedom does it have?



(x, y, z) are inter-related. The two constraints are $z = 0$ and $r^2 - (x^2 + y^2) = 0$.

$\Longrightarrow$ only one independent degree of freedom. We can choose x, but that's awkward. It's even double valued (we can have the same x with a different configuration).

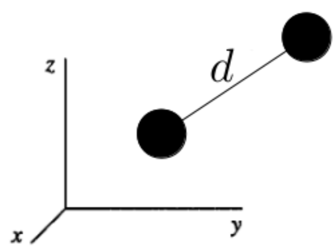
The natural choice is θ: here we only need a single number to determine the location of the pendulum.

Generalized Coordinates are any collection of *independent* variables $(q_1, q_2, ..., q_n)$ (not connected by any equation of constraint) that just suffice to specify uniquely the configuration of a system of particles. The number n of open coordinates is equal to the system's degree of freedom.

For example, if we use:

(i) Smaller # than n coordinates: System's motion is indeterminate
(ii) Larger # than n coordinates: Some coordinates are completely given by others

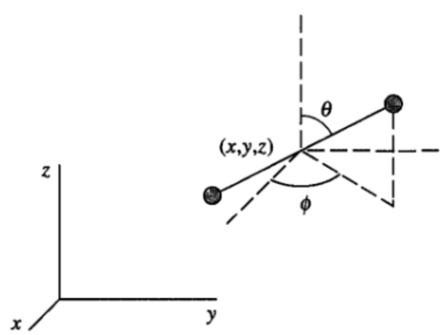
Another Example: Consider two particles connected by a rigid rod.



$$\begin{cases} \text{6 coordinates} : (x_1, y_1, z_1) \quad , \quad (x_2, y_2, z_2) \\ \text{One constraint} : d^2 - [(x_1 - x_2)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] = 0 \end{cases}$$

$\Longrightarrow$ 5 degrees of freedom. Natural choice for generalized coordinates?

$$\boxed{\underbrace{(x, y, z)}_{\text{center of mass}} \text{ and } \underbrace{(\theta, \phi)}_{\text{zenoth and azimuthal}}}$$



In general: N particles require $3N$ coordinates. Suppose there are m constraints:

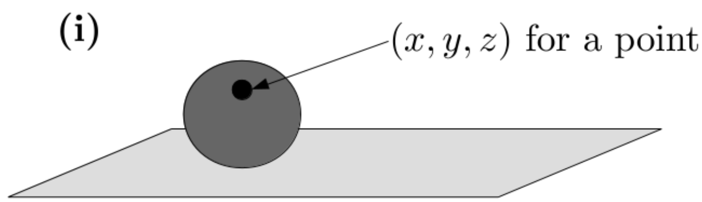
$$f_j = (x_i, y_i, z_i, t) = 0 \qquad j = 1, 2, ..., m \qquad \boxed{\textbf{Holonomic Constraint}}$$

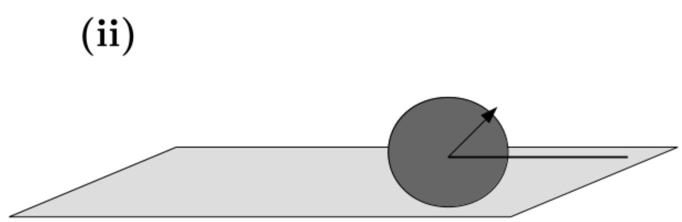
$$\implies \quad (q_1, q_2, ..., q_{3N-m}) \qquad \text{generalized coordinates}$$

Non-holonomic Constrains: e.g $[(x^2 + y^2 + z^2 - R^2)] \geq 0$ (we cannot go inside the earth). This cannot be used to reduce the number of degrees of freedom.

(i) Point in a ball rolling on a table $\implies$ still needs 3 coordinates to describe points; constraint only binds $z \in [0, 2R]$.

(ii) Ball rolling without slipping on a table $\implies$ velocity constraint, not coordinate constraint! (Angular orientation of ball, position in the plane),





$\implies$ There are the coordinates (x, y, z, θ, ϕ) for the ball. We have the holonomic constraint $z = 0$ and the non-holonomic constraint $V_\perp = \sqrt{\dot{x}^2 + \dot{y}^2} = R\dot{\theta}$. Hence there are 4 degrees of freedom.