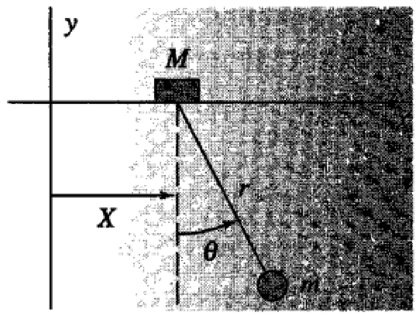
Lecture 27: Generalized Coordinates and Lagrange's Equations of Motion

Calculating T and V in terms of generalized coordinates.

Example: Pendulum attached to a movable support



6 Cartesian Coordinates: (X, Y, Z) and (x, y, z).

4 Holonomic constrains:

$$\begin{cases} Z = 0 \;\; ; \;\; Y = 0 \\ z = 0 \;\; ; \;\; [(x - X)^2 + (y - Y)^2] - r^2 = 0 \end{cases}$$

$\implies$ two generalized coordinates! Choose X and θ. (Note: we can change θ without changing X; therefore θ and X are independent).

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$V = mgy$$

$$\begin{cases} x = X + r\sin\theta \\ y = -r\cos\theta \\ X = X \end{cases} \implies \begin{cases} \dot{x} = \dot{X} + r\dot{\theta}\cos\theta \\ \dot{y} = +r\dot{\theta}\sin\theta \\ \dot{X} = \dot{X} \end{cases}$$

$$T = \frac{1}{2}m\dot{X}^2 + \frac{1}{2}m\left[(\dot{X} + r\dot{\theta}\cos\theta)^2 + (r\dot{\theta}\sin\theta)^2\right]$$

$$= \frac{1}{2}m\dot{X}^2 + \frac{1}{2}m\left[\dot{X}^2 + 2\dot{X}r\dot{\theta}\cos\theta + r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta\right]$$

$$\begin{cases} T = \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m\left[r^2\dot{\theta}^2 + 2r\dot{X}\dot{\theta}\cos\theta\right] \\ \\ V = -mgr\cos\theta \end{cases}$$

Notice how $2r\dot{X}\dot{\theta}\cos\theta$ corresponds to a "cross term." This plays an important role, and it is difficult to get it correctly without starting from Cartesian coordinates.

In addition, note that V only depends on θ and is independent of X! This has important consequences as we will see later.

In general, if there are N particles each with cartesian coordinates $(x_i, y_i, z_i), i = 1, ..., N$ and m holonomic constraints, we can reduce the system to $n = 3N - m$ generalized coordinates (# of degrees of freedom). We specify these as

$$\vec{q} = (q_1, q_2, ..., q_n)$$

We have

$$\begin{cases} x_i = x_i(\vec{q}) \\ y_i = y_i(\vec{q}) \\ z_i = z_i(\vec{q}) \end{cases} \implies \begin{cases} \dot{x}_i = \sum_{j=1}^n \frac{\partial x_i}{\partial q_j}\dot{q}_j &= \dot{x}_i(\vec{q}, \dot{\vec{q}}) \\ \\ \dot{y}_i = \sum_{j=1}^n \frac{\partial y_i}{\partial q_j}\dot{q}_j &= \dot{y}_i(\vec{q}, \dot{\vec{q}}) \\ \\ \dot{z}_i = \sum_{j=1}^n \frac{\partial z_i}{\partial q_j}\dot{q}_j &= \dot{z}_i(\vec{q}, \dot{\vec{q}}) \end{cases}$$

Lagrange's Equations of Motion for a Conservative System

Hamilton's principle:

$$\delta J = \delta\int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i)dt \;=\; \int_{t_1}^{t_2} \delta\mathcal{L}dt \;=\; 0$$

$$= \int_{t_1}^{t_2} \sum_i \left[\frac{\partial\mathcal{L}}{\partial q_i}(\delta q_i) + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}(\delta\dot{q}_i)\right]dt \;=\; 0$$

We integrate the second term by parts. Let $u = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ and $dv = \frac{d(\partial q_i)}{dt}$:

$$\delta(\dot{q}_i) = \frac{d}{dt}(\delta q_i) \implies \int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d(\partial q_i)}{dt}dt \;=\; \frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\delta q_i)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial q_i}\right)(\delta q_i)dt$$

and since

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\delta q_i)\Big|_{t_1}^{t_2} \;=\; 0$$

we have that

$$\delta J = \int_{t_1}^{t_2} dt \sum_i \left[\left(\frac{\partial \mathcal{L}}{\partial q_i}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right](\delta q_i) \;=\; 0$$

(δq_i) is completely arbitrary; the only requirement is that the endpoints are fixed. Consequently, the only way δJ can vanish, given the infinite varieties of $(\delta q)_i$ is that each component variable, i.e,:

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial q_i}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \;=\; 0 \qquad i = 1, ..., n}$$

(Conservative, subject only to holonomic constraints).

n Lagrangian equations of motion for n degrees of freedom.

Applications

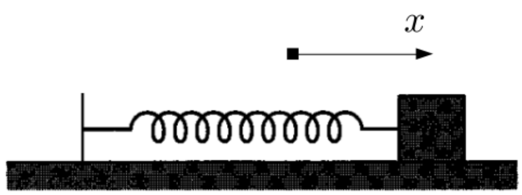
(i): Find suitable set of independent general coordinates q_i

(ii): Find Cartesian coordinates as one of general coordinates: $x_i(\vec{q}), y_i(\vec{q}), z_i(\vec{q})$

(iii): Find T and V as a function of $\vec{q}$, $\vec{q}_i$; Find $\mathcal{L}(\vec{q}, \dot{\vec{q}}) = T - V$

(iv): Use $\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}(\frac{d\mathcal{L}}{d\dot{q}_i}) = 0$ to find the equations of motion.

Example 1: Harmonic Oscillator



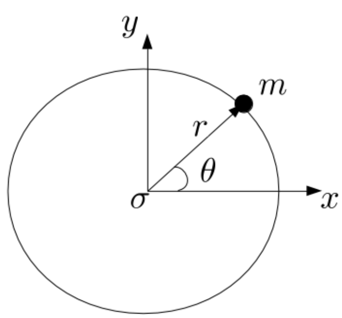
$$\mathcal{L}(x,\dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\left.\begin{array}{l}\frac{\partial\mathcal{L}}{\partial x} = -kx \\ \frac{\partial\mathcal{L}}{\partial \dot{x}} = m\dot{x}\end{array}\right\} \quad \frac{\partial\mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{q}_i}\right) = 0 \implies -kx - \frac{d}{dt}(m\dot{x} = 0)$$

$$\implies -kx - m\ddot{x} = 0$$

$$\implies \boxed{-kx = m\ddot{x}}$$

Example 2: Particle in a Central Force Field



Choose $\vec{q} = (r,\theta)$

$$x = r\cos\theta \implies \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$y = r\sin\theta \implies \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$
$$z = 0 \implies \dot{z} = 0$$

$$\begin{aligned}
T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\
&= \frac{1}{2}\left[\dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\sin^2\theta + \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\cos^2\theta\right] \\
&= \frac{1}{2}m[r^2\dot{\theta}^2 + \dot{r}^2] \quad , \quad \text{and } V = V(r)
\end{aligned}$$

and hence we have

$$\mathcal{L} = \frac{1}{2}m[r^2\dot{\theta}^2 + \dot{r}^2] - V(r)$$

Two equations of motion:

(i): For r

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = 0$$

$$m\dot{\theta}^2 r - \frac{\partial V}{\partial r} - \frac{d}{dt}(m\dot{r}) = 0$$

$$\boxed{m\ddot{r} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2}$$

Note that $-\partial V/\partial r$ is the central force and $mr\dot{\theta}^2$ is the centrifugal force.

(ii): For θ

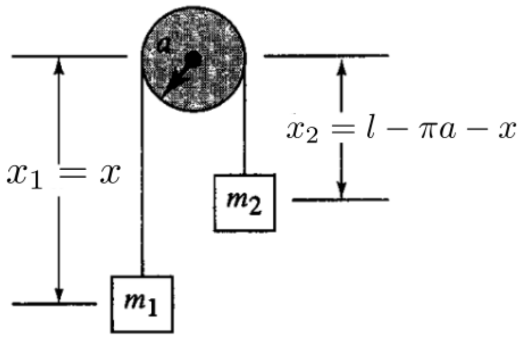
$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0$$

$$0 - \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$\boxed{\frac{d}{dt}(mr^2\dot{\theta}) = 0}$$

Note: Conservation law becomes "automatic"! $\Longrightarrow$ we get a "constant of motion" for each q_i that does not appear explicitly in $\mathcal{L}$.

Example 3: Atwood's Machine



Pulley has moment of inertia I; hence we have two Cartesian coordinates plus the constraint $x_1 + \pi a + x_2 = l$ (l is the length of the cord).

Use $x = x_1$ as coordinate:

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{a}\right)^2 + \frac{1}{2}m_2\dot{x}^2$$

Note that $(\dot{x}/a)^2 = \dot{\theta}^2$. We also have that

$$V = -m_1 g x_1 - m_2 g x_2 = -m_1 g x - m_2 g(l - \pi a - x)$$

$$V = -(m_1 - m_2)gx - m_2 g(l - \pi a)$$

Hence

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}\left(\frac{I}{a^2}\right)\dot{x}^2 + (m_1 - m_2)gx + m_2 g(l - \pi a)$$

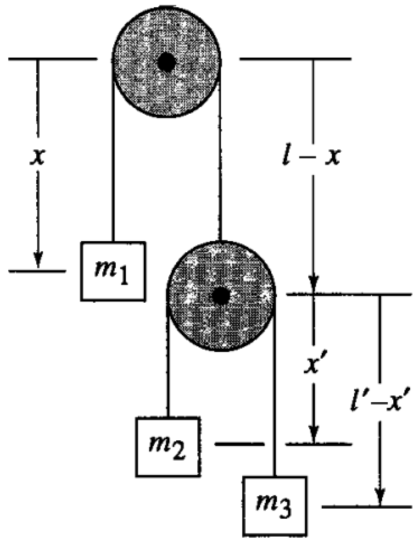
Now

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \left(\frac{\partial \mathcal{L}}{\partial x}\right) \implies \left(m_1 + m_2 + \frac{I}{a^2}\right)\ddot{x} = (m_1 - m_2)g$$

$$\implies \boxed{\ddot{x} = \left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{a^2}}\right)g}$$

We see here that a massive pulley reduces acceleration.

Example 4: The Double Atwood Machine (neglect pulley radii)



Let x_i denote the distance to the mass m_i from the upper pulley and let x_p denote the distance to the pulley which moves (the lower pulley). Let l be the length of the upper rope and l' be the length of the lower rope. Note that

$$\begin{cases} x_1 = x \\ x_2 = (l-x) + x' \\ x_3 = (l-x) + (l'-x') \end{cases}$$

We thus have the following constraints:

$$\begin{aligned} &x_1 + x_p = l \\ &(x_2 - x_p) + (x_3 - x_p) = l' \quad \Longrightarrow \quad x_2 + x_3 - 2(l - x_1) = l' \end{aligned}$$

Now

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(-\dot{x}+\dot{x}')^2 + \frac{1}{2}m_3(-\dot{x}-\dot{x}')^2$$

$$V = -m_1 g x - m_2 g(l - x + x') - m_3 g(l - x + l' - x')$$

$$\Longrightarrow \quad \mathcal{L} \; = \; T - V \; = \; \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 - 2\dot{x}'\dot{x} + \dot{x}'^2) + \frac{1}{2}m_3(\dot{x}^2 + 2\dot{x}\dot{x}' + \dot{x}'^2)$$

$$+(m_1 - m_2 - m_3)gx + (m_2 - m_3)gx' + \text{constant}$$

(i)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \left(\frac{\partial \mathcal{L}}{\partial x}\right)$$

$$\boxed{(m_1 + m_2 + m_3)\ddot{x} + (m_3 - m_2)\ddot{x}' = (m_1 - m_2 - m_3)g}$$

(ii)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}'}\right) = \left(\frac{\partial \mathcal{L}}{\partial x'}\right)$$

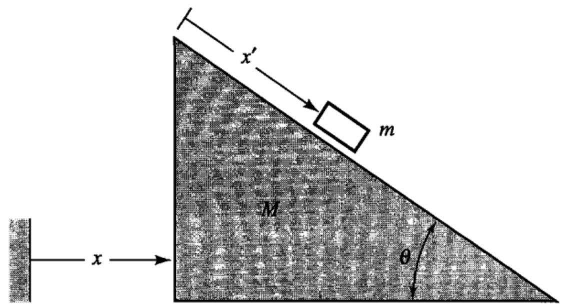
$$\boxed{(m_2 + m_3)\ddot{x}' + (m_3 - m_2)\ddot{x} = (m_2 - m_3)g}$$

We obtain the accelerations by solving this system of two equations and two unknowns ($\ddot{x}$ and $\ddot{x}'$)

$$\Longrightarrow \quad \boxed{\ddot{x} = -g\left[\frac{(m_3 - m_2)^2 + (m_2 + m_3)(m_1 - m_2 - m_3)}{(m_3 - m_2)^2 - (m_2 + m_3)(m_1 + m_2 + m_3)}\right]}$$

In the homework, be wary of the "double-double" Atwood machine (m_1 is replaced by a pulley which itself holds two masses).

Example 5: Particle Sliding on a movable inclined plane



Let $\vec{V}$ be the velocity of the inclined plane and $\vec{v}$ be the velocity of the block. Then

$$\vec{V} = \hat{i}\dot{x}$$
$$\vec{v} = \hat{i}\dot{x} + \hat{e}_\theta \dot{x}'$$

where $\hat{e}_\theta = (\cos\theta, -\sin\theta)$. We thus have

$$\begin{aligned} T &= \frac{1}{2}MV^2 + \frac{1}{2}mv^2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\hat{i}\dot{x} + \hat{e}_\theta \dot{x}')^2 \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + \cos\theta\dot{x}')^2 + (\sin\theta\dot{x}')^2\right] \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2\cos\theta\dot{x}\dot{x}' + \cos^2\theta\dot{x}'^2 + \sin^2\theta\dot{x}'^2\right] \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2\cos\theta\dot{x}\dot{x}' + \dot{x}'^2\right] \end{aligned}$$

and

$$V = mg[L\sin\theta - x'\sin\theta] = (\text{constant}) - mg\sin\theta x'$$

where L is the length of the plane. Hence

$$\mathcal{L} = T - V = \frac{1}{2}(M+m)\dot{x}^2 + m\cos\theta\dot{x}\dot{x}' + \frac{1}{2}m\dot{x}'^2 + mg\sin\theta x'$$

Equations of Motion:

(i)

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)=\left(\frac{\partial\mathcal{L}}{\partial x}\right)$$
$$\implies \quad \frac{d}{dt}((M+m)\dot{x}+m\cos\theta\dot{x}')=0$$

(since $\mathcal{L}$ is independent of x). Note that conservation of momentum comes out automatically.

(ii)

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}'}\right)=\left(\frac{\partial\mathcal{L}}{\partial x'}\right)$$
$$\implies \quad \frac{d}{dt}(m\cos\theta\dot{x}+m\dot{x}')=mg\sin\theta$$

We now have two equations and two unknowns:

$$\begin{cases}(M+m)\ddot{x}+m\cos\theta\ddot{x}'=0\\ \cos\theta\ddot{x}+\ddot{x}'=g\sin\theta\end{cases}$$

Solving for $\ddot{x}$ and $\ddot{x}'$ yields

$$\boxed{\ddot{x}'=\frac{g\sin\theta}{1-\frac{m}{m+M}\cos^2\theta}} \qquad \boxed{\ddot{x}=-\frac{g\sin\theta\cos\theta}{\frac{m+M}{m}-\cos^2\theta}}$$