Lecture 29

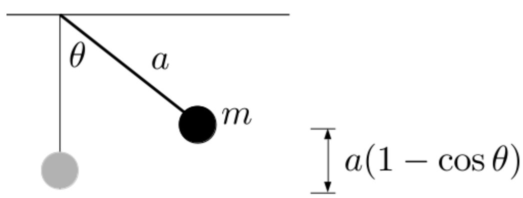
Generalized Forces

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \Longrightarrow \quad \dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

where p_i is the generalized momentum and $\partial \mathcal{L}/\partial q_i$ is the general force. The second equation is Lagrange's equation.

$$\text{Particles in 1D:} \quad \mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$$
$$F_x = \frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial V}{\partial x}$$

Pendulum:



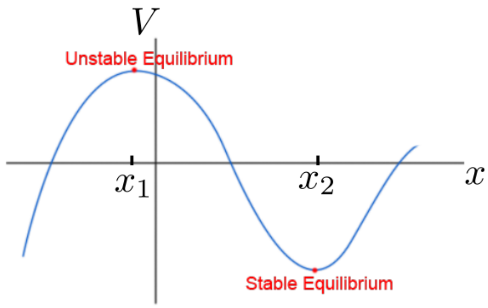
$$\mathcal{L} = \frac{1}{2}m(a\dot{\theta})^2 - mga(1-\cos\theta))$$

$$F_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = -mga\sin\theta \quad = \quad \text{Torque}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ma^2\dot{\theta} = I\omega \quad = \quad \text{Angular Momentum}$$

$$\dot{p}_\theta = F_\theta \quad \Longrightarrow \quad \boxed{I\alpha = \tau}$$

Equilibrium and Stability



x_1 and x_2 are equilibrium points since $\partial V/\partial x|_{x=x_1,x_2}=0$

x_1 is unstable since $\partial^2 V/\partial x^2|_{x=x_1} < 0$

x_2 is stable since $\partial^2 V/\partial x^2|_{x=x_2} > 0$

We wish to generalize to a single coordinate q:

$$V(q) = V_0 + V_0' q + \frac{1}{2} V_0'' q^2 + ... + \frac{1}{n!} V_0^{(n)} q^n + ...$$

where $V_0 = V(q=0)$ and $V_0' = dV/dq|_{q=0}$ and so on. If $q=0$ is a point of equilibrium, $V_0' = 0$. (If it is not, we can define $q' = q - q_0$ where q_0 is the equilibrium point and repeat the argument).

Consequently, for small q we may approximate (Set $V_0 = 0$)

$$V(q) \approx \frac{1}{2} V_0'' q^2$$

$$\text{Provided } V_0'' \neq 0 \qquad \begin{cases} V_0'' > 0 & \text{stable} \\ V_0'' < 0 & \text{unstable} \end{cases}$$

What if $V_0'' = 0$? Then check the 1st non-vanishing $V_0^{(n)}$. If n is odd, e.g.,

$$V(q) = \frac{1}{3!} V_0''' q^3 \qquad F(q) = -\frac{\partial V}{\partial q} = -\frac{1}{2!} V_0''' q^2 \quad \Longrightarrow \quad \text{unstable}$$

If m is even and $V_0^{(n)} > 0$, then the particle is stable.

For n degrees of freedom, assume $q_1 = q_2 = ... = q_n = 0$ is the equilibrium point; expand $V(\vec{q})$:

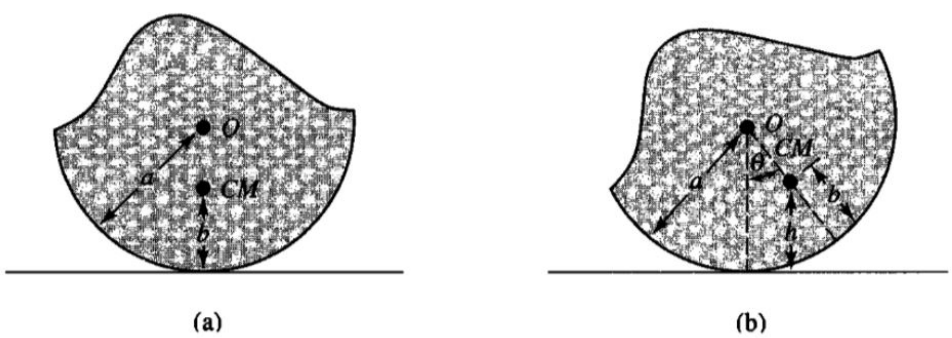
$$V = V(q_1, q_2, ..., q_n) = \frac{1}{2}(k_{11}q_1^2 + k_{12}q_1q_2 + k_{21}q_2q_1 + k_22q_2^2 + ...)$$

where $k_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$. The equation above is a "quadratic form."

$\vec{q}$ is stable if the quadratic form is positive definite, i.e. $V = 0$ or $V > 0$ for all $\vec{q}$. We may show this is true if

$$k_{11} > 0 \quad , \quad k_{22} > 0 \quad \text{and} \quad \begin{vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{vmatrix} > 0 \quad , \quad \text{and so on}$$

Example: Stability of round-bottomed objects



O is the center of curvature. Note that $h = a - (a - b)\cos\theta$. We have that

$$V = mgh = mga - mg(a - b)\cos\theta$$

$$F = -\frac{\partial V}{\partial \theta} = -mg(a - b)\sin\theta \quad \Longrightarrow \quad \theta = 0 \;\; \text{is an equilibrium point}$$

$$V_0'' = \frac{\partial^2 V}{\partial \theta^2}\bigg|_{\theta=0} = mg(a-b)\cos\theta\Big|_{\theta=0} = mg(a-b) \quad \Longrightarrow \quad \begin{cases} a > b & \Longrightarrow \quad \text{stable} \\ a < b & \Longrightarrow \quad \text{unstable} \\ a = b & \Longrightarrow \quad \text{neutral} \end{cases}$$

Small Oscillations: Suppose we have stability and $a > b$. Then for $\theta << 1$

$$V(\theta) \approx \frac{1}{2}V_0''\theta^2 = \frac{1}{2}mg(a-b)\theta^2$$

$T =?$. Assume the object rolls without slipping $\implies v_{cm} = b\dot{\theta}$ for $\theta << 1$. Then

$$T = \frac{1}{2}m(b\dot{\theta})^2 + \frac{1}{2}I_{cm}\dot{\theta}^2$$

It follows that

$$\mathcal{L} = T - V \;\; = \;\; \frac{1}{2}mb^2\left[1 + \frac{I_{cm}}{mb^2}\right]\dot{\theta}^2 - \frac{1}{2}mg(a-b)\theta^2$$

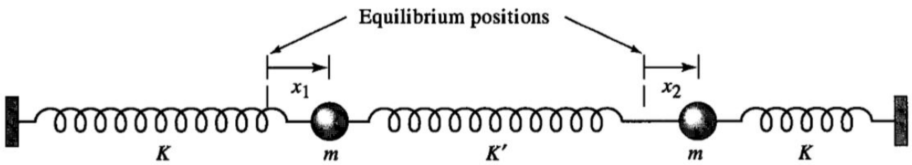
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} \quad \implies \quad mb^2\left(1 + \frac{I_{cm}}{mb^2}\right)\ddot{\theta} = -mg(a-b)\theta$$

We call $mb^2(1 + \frac{I_{cm}}{mb^2})$ the m_{eff}. Recall that $mg(a-b) = V_0''$. Our differential equation becomes

$$\ddot{\theta} + \frac{V_0''}{m_{eff}}\theta = 0$$

It follows that $\omega_{osc} = \sqrt{V_0''/m_{eff}}$ is the frequency of "small oscillation."

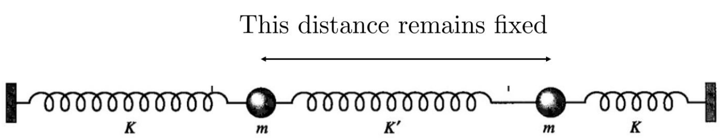
Coupled Harmonic Oscillators: Normal Modes or Collective Normal Modes of Oscillation



(i): Suppose $x_1(t=0) > 0$ amd $x_2(0) = 0$. m_1 will oscillate, and since it is coupled to the second mass, part of its energy will be transfered to m_2. Energy will flow back and forth! Each individual mass cannot oscillate alone!

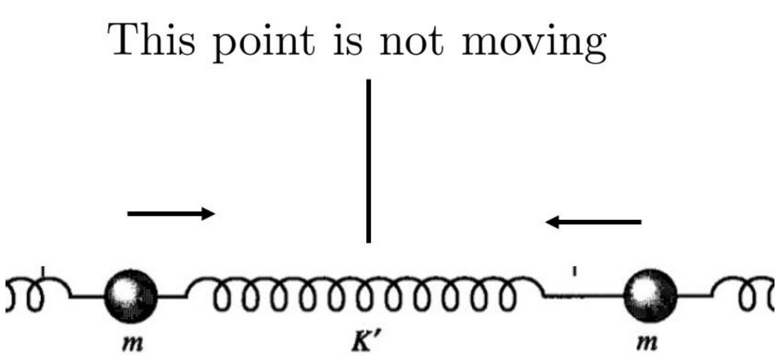
(ii): Consider displacing $x_1 = x_2 = x$. The spring K' is not compressed/stretched, hence the restoring force is $2Kx$. Since the total mass is $2m$, the frequency is

$$\omega_1 = \sqrt{\frac{2K}{2m}} = \sqrt{\frac{K}{m}}$$



This is called a "normal mode" or "collective mode."

(iii): Another initial condition where $x_1 = -x_2$ is called "breathing mode." In this situation, the center of the of the middle spring remains fixed while m_1 and m_2 move in and out together.



Find frequency? Restoring force on *each* mass is $F = -kx_1 - 2K'x_1 = -(K+2K')x$. Note that $2K$ is the K_{eff} of the middle spring cut in half (we are exploiting symmetry here). Hence

$$\omega_2 = \sqrt{\frac{K+2K'}{m}}$$

All possible motions are linear combinations of ω_1 and ω_2.

Method of Solution

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}Kx_1^2 - \frac{1}{2}K'(x_1-x_2)^2 - \frac{1}{2}kx_2^2$$

Using $\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{q}}) = \frac{\partial\mathcal{L}}{\partial q}$ gives us

$$\begin{aligned} m\ddot{x}_1 &= -Kx_1 - K'(x_1-x_2) \\ m\ddot{x}_2 &= K'(x_1-x_2) - Kx_2 \end{aligned}$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \ddot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} \quad = \quad - \begin{pmatrix} +(K+K') & -K \\ -K' & (K+K') \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or, in matrix form, $\mathbb{M}\cdot\ddot{\vec{q}} = -\mathbb{K}\cdot\vec{q}$. We search for "normal mode solutions" of the form $\vec{q} = \vec{a}\cos(\omega t - \delta)$. Since $\ddot{\vec{q}} = -\omega^2\vec{q}$:

$$-\omega^2\mathbb{M}\cdot\vec{q} = -\mathbb{K}\cdot q \quad , \quad \text{divide by} \cos(\omega t - \delta)$$

$$\implies \quad (\mathbb{K} - \omega^2\mathbb{M})\cdot\vec{a} = 0$$

This is a homogeneous system, non-zero solutions exist only when $\det(\mathbb{K} - \omega^2\mathbb{M}) = 0$. If $\mathbb{M}$ is diagonal, our problem is to find the eigenvalues ω^2 and eigenvectors $\vec{a}$ of $\mathbb{K}$.

$$\begin{vmatrix} (K+K') - m\omega^2 & -K' \\ -K' & (K+K') - m\omega^2 \end{vmatrix} \; = \; 0 \implies [(K+K') - m\omega^2]^2 - K'^2 = 0$$

$$(K+K') - m\omega^2 = \pm K \quad \implies \quad \omega^2 = \frac{K+K' \pm K'}{m} \quad = \quad \begin{cases} \omega_- = \omega_1 = \frac{k}{m} \\ \omega_+ = \omega_2 = \frac{2K'+K}{m} \end{cases}$$

Eigenvectors $\vec{a}$? Plug $w_1^2 = k/m$:

$$\begin{pmatrix} (K+K') - m\omega_1^2 & -K' \\ -K' & (K+K') - m\omega_1^2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \quad K' \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \quad a_{11} - a_{21} = 0 \implies a_{11} = a_{21} \implies \vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A$$

Similarly with $\omega_2^2 = (2K' + K)/m$ we get

$$\begin{pmatrix} (K+K') - (2K'+K)' & -K' \\ -K' & (K+K') - (2K'+K) \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \quad K' \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \quad a_{11} = -a_{22} \implies \vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} B$$

Therefore the general solution is given by

$$\vec{q} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\cos(\omega_1 t + \delta_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\cos(\omega_2 t + \delta_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Two Normal Modes: 4 constants of motion, determined by the initial conditions $x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)$

Another simple way to solve the problem is to transform to *normal coordinates*:

$$\begin{cases} \ddot{x}_1 + (\frac{K+K'}{m})x_1 - (\frac{K'}{m})x_2 = 0 & \textbf{(1)} \\ \ddot{x}_2 + (\frac{K+K'}{m})x_2 - (\frac{K'}{m})x_1 = 0 & \textbf{(2)} \end{cases}$$

Add and subtract:

$$\begin{aligned} (1)+(2) \quad \Longrightarrow \quad (x_1 \ddot{+} x_2) + \left[\left(\frac{K+K'}{m}\right) - \frac{K'}{m}\right](x_1+x_2) &= 0 \\ (1)-(2) \quad \Longrightarrow \quad (x_1 \ddot{+} x_2) + \left[\left(\frac{K+K'}{m}\right) + \frac{K'}{m}\right](x_1+x_2) &= 0 \end{aligned}$$

Therefore when transforming to $Q_1 = x_1 + x_2$ and $Q_2 = x_1 - x_2$ we get

$$\begin{cases} \ddot{Q}_1 + \frac{K}{m}Q_1 = 0 \\ \ddot{Q}_2 + \frac{2K'+K}{m}Q_2 = 0 \end{cases}$$

These are known as decoupled equations.

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \Longrightarrow \quad \vec{x} = \mathbb{O}\cdot\vec{Q}$$

Actually we may show that this transform matrix equals

$$\mathbb{O} = (\vec{a}_1\vec{a}_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Proof: $\vec{a}_1 = \mathbb{O}\cdot\binom{1}{0}$ and $\vec{a}_2 = \mathbb{O}\cdot\binom{0}{1}$ where

$$\mathbb{M}^{-1}\cdot\mathbb{K}\cdot\vec{a}_1 = \frac{k}{m}\vec{a}_1 \quad \Longrightarrow \quad \mathbb{M}^{-1}\cdot\mathbb{K}\cdot\mathbb{O}\cdot\binom{1}{0} = \frac{k}{m}\mathbb{O}\cdot\binom{1}{0}$$

$$\begin{aligned} &\Longrightarrow (1\ 0)(\mathbb{O}^{-1}(\mathbb{M}\cdot\mathbb{K})\mathbb{O})\binom{1}{0} = \frac{k}{m} \\ &\Longrightarrow (0\ 1)(\mathbb{O}^{-1}(\mathbb{M}\cdot\mathbb{K})\mathbb{O})\binom{1}{0} = 0 \end{aligned}$$

The same goes for $\vec{a}_2$. Therefore $\mathbb{O} = (\vec{a}_1\ \vec{a}_2)$

Now

$$\mathbb{M}\cdot\ddot{\vec{q}} = -\mathbb{K}\cdot\vec{q} \quad\Longrightarrow\quad \ddot{\vec{q}} = -\mathbb{M}^{-1}\cdot\mathbb{K}\cdot\vec{q}$$

$$\Longrightarrow\quad \ddot{(\mathbb{O}^{-1}\cdot\vec{q})} = -\mathbb{O}^{-1}\cdot(\mathbb{M}^{-1}\cdot\mathbb{K})\cdot\mathbb{O}\cdot(\mathbb{O}^{-1}\cdot\vec{q})$$

$$\Longrightarrow\quad \begin{pmatrix}\ddot{Q}_1\\ \ddot{Q}_2\end{pmatrix} \quad=\quad -\begin{pmatrix}k/m & 0\\ 0 & (2K'+K)/m\end{pmatrix}\cdot\begin{pmatrix}Q_1\\ Q_2\end{pmatrix}$$

The reference system of the normal coordinates "decouples" the problem.