

## Lecture 30

**Generalization:** Lagrangian close to an equilibrium point  $\vec{q} = \vec{0}$ :

$$\mathcal{L} = \frac{1}{2} \sum_{ij} M_{ij} \dot{q}_i \dot{q}_j - \frac{1}{2} \sum_{ij} K_{ij} q_i q_j$$

If  $M_{ij}$  and  $K_{ij}$  originally depend on  $q_i$ , approximate them as  $M_{ij}(q_i) \approx M_{ij}(q_i = 0)$  (same for K).

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i} &\implies \sum_j M_{ij} \ddot{q}_j = - \sum_j K_{ij} q_j \quad i = 1, 2, \dots, n \\ &\implies \mathbb{M} \cdot \ddot{\vec{q}} = -\mathbb{K} \cdot \vec{q} \end{aligned}$$

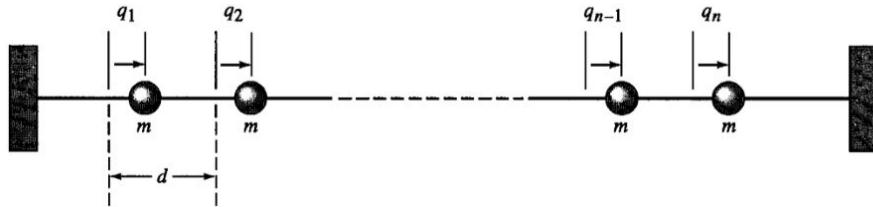
If a solution of the form  $\vec{q} = \vec{a} \cos(\omega t - \delta)$  exists then

$$\begin{aligned} (\mathbb{K} - \omega^2 \mathbb{M}) \cdot \vec{a} &= \vec{0} \\ \implies (\mathbb{M}^{-1} \cdot \mathbb{K} - \omega^2 \mathbb{I}) \cdot \vec{a} &= \vec{0} \end{aligned}$$

$n$  eigenvalues  $w_i^2$  and  $n$  eigenvectors  $\vec{a}_i$ :

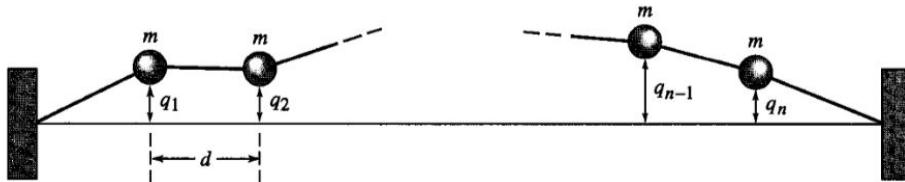
$$\boxed{\vec{q}(t) = \sum_{i=1}^n A_i \cos(\omega_i t - \delta_i) \vec{a}_i}$$

**Example:** Linear Array of Coupled Harmonic Oscillators



**Longitudinal Oscillations:**

$$V(\vec{q}) = \sum_{j=0}^n \frac{1}{2} K_L (q_{j+1} - q_j)^2$$



**Transverse Oscillations**

$$\text{Work done by string} = \sqrt{d^2 + (q_{j+1} - q_j)^2} = d \left[ 1 + \frac{1}{2d^2} (q_{j+1} - q_j)^2 \right] = d + \frac{1}{2d} (q_{j+1} - q_j)^2$$

$$T\Delta l = \frac{T}{2d} (q_{j+1} - q_j)^2$$

$$\Rightarrow V(\vec{q}) = \sum_j \frac{1}{2} K_T (q_{j+1} - q_j)^2 \quad (K_T = T/d)$$

For both cases,

$$\mathcal{L} = \frac{1}{2} \sum_{j=0}^n [m\dot{q}_j^2 - K(q_{j+1} - q_j)^2]$$

and  $q_0 = q_n + 1 = 0$  (fixed endpoints).

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i} \implies m\ddot{q}_j = \frac{\partial}{\partial q_j} \left[ -\frac{K}{2}(q_{j+1} - q_j)^2 - \frac{K}{2}(q_j - q_{j-1})^2 \right]$$

$$m\ddot{q} = +K(q_{j+1} - q_j) - K(q_j - q_{j-1})$$

$$\implies m\ddot{q}_j - Kq_{j+1} + 2Kq_j - Kq_{j-1} = 0$$

$$\begin{pmatrix} 2K - m\omega^2 & -K & 0 & \dots & 0 \\ -K & 2K - m\omega^2 & -K & \dots & 0 \\ 0 & -K & 2K - m\omega^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2K - m\omega^2 \end{pmatrix} \cdot \vec{q} = \vec{0}$$

This is hard to diagonalize so we try the ansatz:

$$q_j^{(k)}(t) = A \sin(kj) \cos(\omega t)$$

$k$  labels the normal anodes. Later, we will find the allowed  $k$ 's.

$$\implies -m\omega^2 \sin(kj) - K \sin[k(j+1)] + 2K \sin(kj) - K \sin[k(j-1)] = 0$$

Using the trig identity  $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$  we simplify the expression above:

$$[-m\omega^2 - 2K \cos(k) + 2K] = 0$$

$$\frac{m\omega^2}{K} - 2(1 - \cos k) = 0 \implies w_k^2 = \frac{4K}{m} \sin^2 \left( \frac{k}{2} \right)$$

$$\implies w_k = 2\sqrt{\frac{K}{m}} \left| \sin \left( \frac{k}{2} \right) \right|$$

Boundary Conditions:  $\begin{cases} q_{j=0}^{(k)} = 0 \implies q_j(t) = A \sin(kj) \cos(\omega t) & \text{satisfied for all k's} \\ q_{j=n+1}^{(k)} = 0 \implies \sin[k(n+1)] = 0 \implies k = \left(\frac{N\pi}{n+1}\right), \quad N = 1, 2, \dots, n \end{cases}$

$$\boxed{\text{N-th normal mode} \implies q_j^{(n)}(t) = A \sin\left(\frac{j\pi N}{n+1}\right) \cos(\omega t)}$$

Note that

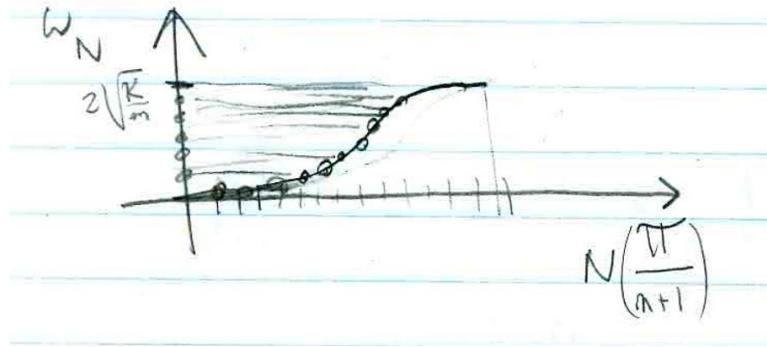
$$\begin{aligned} N = (n+1) &\implies q_j^{N=n+1} \propto \sin(j\pi) = 0 \\ N = (n+2) &\implies q_j^{N=n+2} = A \sin\left(j\pi + \frac{j\pi}{n+1}\right) = (-1)^j A \sin\left(\frac{j\pi}{n+1}\right) \end{aligned}$$

Compare to

$$q_j^{N=n} = A \sin\left(j\pi - \frac{j\pi}{n_1}\right) = (-1)^j A \sin\left(-\frac{j\pi}{n+1}\right) = -q_j^{N=n+2}$$

Hence  $N = n+2$  is the same as  $N = n$ . Now

$$\omega_N = 2\sqrt{\frac{K}{m}} \left| \sin\left(N \frac{\pi}{2(n+1)}\right) \right|$$



Also note

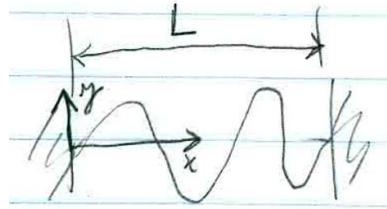
$$q_j = A \sin\left(\frac{j\pi N}{n+1}\right) = A \sin\left(\frac{(jd)2\pi}{2(n+1)d/N}\right) = A \sin\left(\frac{2\pi x}{\lambda_1}\right)$$

$$\implies \lambda_N = \frac{2L}{N} \implies \boxed{\frac{N\lambda}{2} = L}$$

“Continuum” limit ( $n \rightarrow \infty$ ,  $d \rightarrow 0$  with  $nd = L$  finite)

Since  $K_T = T/d$  we have

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} \sum_{j=0}^n \left[ \frac{dm}{d} \dot{q}_j^2 - \frac{d}{d} \left( \frac{T}{d} \right) (q_{j+1} - q_j)^2 \right] \\ &= \frac{1}{2} \int_0^L dx \left[ \mu (\dot{y}(x))^2 - T \left( \frac{y(x+d) - y(x)}{d} \right)^2 \right] \\ &= \frac{1}{2} \int_0^L dx \left[ \mu \left( \frac{\partial y}{\partial t}(x, t) \right)^2 - T \left( \frac{\partial y}{\partial x} \right)^2 \right]\end{aligned}$$



The term being integrated is known as the “Lagrangian density.”

**Action:**

$$J = \int \mathcal{L} dt = \frac{1}{2} \int dt \int dx [\mu \dot{y}^2 - T(y')^2] = J[y(x, t)]$$

Hence we have

$$\delta J = \frac{1}{2} \int dt \int dx [2\mu \dot{y} \delta(\dot{y}) - 2T(y') \delta(y')]$$

and since  $\dot{y} \delta(y) = -\ddot{y} \delta(y)$  and  $y' \delta(y') = -y'' \delta(y')$  we have

$$\begin{aligned}\delta J &= \int dt \int dx [-\mu \ddot{y} + Ty''](\delta y) \\ \implies \boxed{-\mu \frac{\partial^2 y}{\partial t^2} + T \frac{\partial^2 y}{\partial x^2} = 0} \quad \boxed{v = \sqrt{\frac{T}{\mu}}}\end{aligned}$$

This is the wave equation!

$$\boxed{\frac{\partial^2 y(x,t)}{\partial t^2} - v^2 \frac{\partial^2 y(x,t)}{\partial x^2} = 0 \quad ; \quad v = \sqrt{\frac{T}{\mu}}}$$

The solution is any function  $y(x,t) = f(x \pm vt)$ . Note that  $\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t} \frac{\partial d}{\partial x'} = \pm v \frac{\partial}{\partial x'}$  and  $\frac{\partial}{\partial x} = \frac{\partial x}{\partial x'} \frac{\partial}{\partial x'} x' = \frac{\partial}{\partial x'}$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} \left[ \pm v \frac{\partial f}{\partial x'} \right] = \pm v \frac{\partial}{\partial x'} \left[ \pm v \frac{\partial f}{\partial x'} \right] = v^2 \frac{\partial^2 f}{\partial x'^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} - v^2 \frac{\partial^2 f}{\partial x^2} = \left( v^2 \frac{\partial^2 f}{\partial x^2} \right) - v^2 \frac{\partial^2 f}{\partial x^2} = 0$$

