P423 - Lecture 4 : Gauge transformations and Aharonov-Bohm effect

We have seen that the Hamiltonian of a particle in a $\vec{B}$ field is $H = \frac{1}{2m}(\vec{p} - q\vec{A})^2$. But how does the wave func depends on $\vec{A}$? Shouldn't it depend only on $\vec{B}$? Quantum mechanics gives rise to another surpris

review Gauge transformation

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A} \quad (\text{Because } \vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0)$$

Maxwell eqns

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \Rightarrow \quad \vec{\nabla}\times\left[\vec{E} + \frac{\partial\vec{A}}{\partial t}\right] = \vec{0} \qquad \left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) \text{ is conservative}$$

allows the definition of an electric potential Φ :

$$\Rightarrow \quad \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\Phi$$

Relation between fields and potentials :

$$\begin{cases} \vec{E} = -\vec{\nabla}\Phi - \dfrac{\partial\vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$$

But the prescription $\vec{B} = \vec{\nabla}\times\vec{A}$ allows some freedom in choosing $\vec{A}$. Common choices : Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$; Lorentz gauge $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial\Phi}{\partial t}$

Define a new gauge : $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \Rightarrow$

$$\begin{cases} \vec{E} = -\vec{\nabla}\Phi - \dfrac{\partial}{\partial t}\left[\vec{A}' - \vec{\nabla}\Lambda\right] = -\vec{\nabla}\underbrace{\left[\Phi - \dfrac{\partial\Lambda}{\partial t}\right]}_{\Phi'} - \dfrac{\partial\vec{A}'}{\partial t} \\ \vec{B} = \vec{\nabla}\times\left[\vec{A}' - \vec{\nabla}\Lambda\right] = \vec{\nabla}\times\vec{A}' \end{cases}$$

Hence a gauge transformation is defined as

$$\begin{cases} \vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \\ \Phi \rightarrow \Phi' = \Phi - \dfrac{\partial\Lambda}{\partial t} \end{cases} \qquad \text{where } \Lambda = \Lambda(\vec{r}, t) .$$

②

Schrödinger eqn for a charged particle in an electromagnetic field:

$$\left\{ \frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla} - q\vec{A}(\vec{r},t)\right]^2 + q\,\Phi(\vec{r},t)\right\}\psi(\vec{r},t) = i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}$$

We will now show that under a gauge transformation the wave fnc changes into

$$\psi(\vec{r},t) \to \psi'(\vec{r},t) = e^{\frac{iq}{\hbar}\Lambda(\vec{r},t)}\psi(\vec{r},t)$$

Multiply the Schrödinger eqn by $e^{\frac{iq}{\hbar}\Lambda}$ on both sides and use

$$e^{f(x)}\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\right)e^{f(x)} \Rightarrow e^{\frac{iq\Lambda}{\hbar}}\left[\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right]\psi = \left[\frac{\hbar}{i}\vec{\nabla} - q\vec{A} - q\nabla\Lambda\right]e^{\frac{iq\Lambda}{\hbar}}\psi$$

So that

$$\left\{\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla} - q\left(\vec{A}+\vec{\nabla}\Lambda\right)\right]^2 + q\Phi\right\}e^{\frac{iq}{\hbar}\Lambda}\psi = i\hbar\left[\frac{\partial}{\partial t} + \frac{q}{i\hbar}\frac{\partial\Lambda}{\partial t}\right]e^{\frac{iq\Lambda}{\hbar}}\psi$$

$$\Rightarrow \left\{\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla} - q(\vec{A}+\vec{\nabla}\Lambda)\right]^2 + q\left(\Phi - \frac{\partial\Lambda}{\partial t}\right)\right\}\left[e^{i\frac{q\Lambda}{\hbar}}\psi\right] = i\hbar\frac{\partial}{\partial t}\left[e^{i\frac{q\Lambda}{\hbar}}\psi\right]$$

or

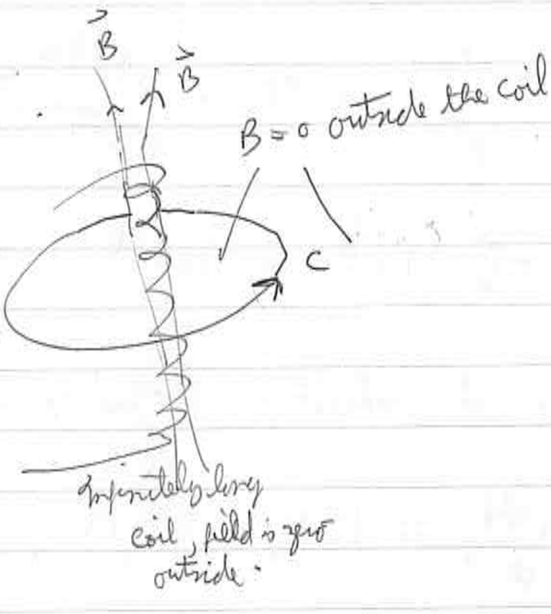
$$\left\{\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla} - q\vec{A}'\right]^2 + q\,\Phi'\right\}\psi' = i\hbar\frac{\partial\psi'}{\partial t} \quad //$$

which is the Schrödinger eqn in the primed potentials.

Hence under gauge transf. the wave fnc acquires a phase $\Rightarrow |\psi|^2 = |\psi'|^2$, but in QM phase differences can be observable.

Aharonov-Bohm Effect

Assume $\vec{B} \neq 0$ in a finite region of space. In this case you need $\vec{A} \neq 0$ all over the space. Because even in the regions where $B = 0$ we get



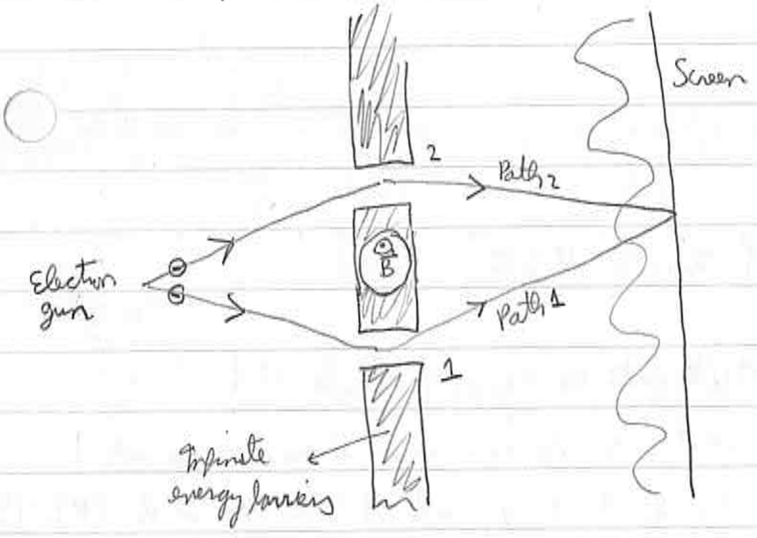
$$\oint_C \vec{A} \cdot d\vec{r} = \int_S \vec{\nabla} \times \vec{A} \cdot d\hat{a} = \int_S \vec{B} \cdot d\hat{a} = \Phi_B \neq 0$$

$\vec{A}$ for $\vec{r} \in B = 0$ region!

For $\vec{r}$ in the $B = 0$ region we have $\vec{A} = \vec{\nabla} \Lambda$, with $\Lambda(\vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \vec{A} \cdot d\vec{r}$.

Consider the interference experiment: The particles can not penetrate the region with $B > 0$.

The particles only travel in the $B = 0$ region.



Let's calculate the interference pattern: Turning on B inside the barrier is like doing a gauge transformation $\vec{A} \to \vec{A}' + \nabla \Lambda$:

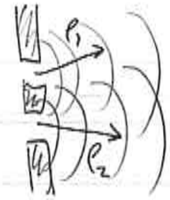
$$\psi_{Path\,1}(\vec{r}) = \psi_1(\vec{r})\, e^{\frac{iq}{\hbar}\int_1 \vec{A} \cdot d\vec{r}}$$

$$\psi_{Path\,2}(\vec{r}) = \psi_2(\vec{r})\, e^{\frac{iq}{\hbar}\int_2 \vec{A} \cdot d\vec{r}}$$

$$\psi_{Screen}(\vec{r}) = \psi_1 e^{\frac{iq}{\hbar}\int_1 \vec{A} \cdot d\vec{r}} + \psi_2 e^{\frac{iq}{\hbar}\int_2 \vec{A} \cdot d\vec{r}} = \left[\psi_1 e^{\frac{iq}{\hbar}\underbrace{\left(\int_1 - \int_2\right)\vec{A} \cdot d\vec{r}}_{= \oint \vec{A} \cdot d\vec{r} \,=\, \Phi_B \text{ flux} \neq 0!}} + \psi_2\right] e^{\frac{iq}{\hbar}\int_2 \vec{A} \cdot d\vec{r}}$$

$$= \left[\psi_1 e^{\frac{iq}{\hbar}\Phi_B} + \psi_2\right] e^{\frac{iq}{\hbar}\int_2 \vec{A} \cdot d\vec{r}}$$

If we consider ψ_1 and ψ_2 as free particle states



$\rho_1 = |\vec{r} - \vec{r}_{slit\,1}|$

$\rho_2 = |\vec{r} - \vec{r}_{slit\,2}|$

$$\psi_1 = \frac{e^{ik\rho_1}}{\sqrt{\rho_1}}, \quad \psi_2 = \frac{e^{ik\rho_2}}{\sqrt{\rho_1}}$$

(asymptotic solutions of $-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$ in cylindrical coordinates, for $\rho = \sqrt{x^2+y^2} \to \infty$)

Then we will get constructive interference when:

$$k\rho_1 + \frac{q\Phi_B}{\hbar} = k\rho_2 + 2\pi m, \quad \text{or}$$

$$\rho_1 - \rho_2 = \underbrace{\frac{\lambda}{2\pi}}_{\frac{1}{k}}\left(2\pi m - \frac{q}{\hbar}\Phi_B\right)$$

The positions of the interference maxima will shift due to $B > 0$,

even though the electrons never penetrate into the region of finite B!

⇒ It is as if the wave func senses all the space around it, even the regions where it can not go into!

Demonstrated experimentally in R. C. Chambers, PRL 5, 3 (1960); and A. Tonomura et al, PRL 48, 1443 (1982).

→ Show slides about SQUID and DWave.

~ 30 mins.