P423: Lecture 7: Addition of angular momenta

Fine structure of H-atom: E_n + 1st order perturbation theory

$$E_{nj} = -\frac{R_y}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right] \qquad \left(\alpha \simeq \frac{1}{137.06} \text{ fine structure const.}\right)$$

j is the total angular momentum eigenvalue: $(\vec{J})^2 |j\, m_j\rangle = j(j+1)\hbar^2 |j\, m_j\rangle$.

Addition of angular momen.

$$\vec{J} = \underset{\substack{\downarrow \\ l}}{\vec{L}} + \underset{\substack{\downarrow \\ \frac{1}{2}}}{\vec{S}} \quad \Rightarrow \begin{cases} j = l + \frac{1}{2} \\ j = l - \frac{1}{2} \end{cases}$$

E.g. for $j = l + \frac{1}{2}$:

$$|j = l + \tfrac{1}{2},\ m_j = l + \tfrac{1}{2}\rangle = |l, m_l = l\rangle |\uparrow\rangle$$

Easy! That's the only product with $|l\ m_l\rangle$ and $|\uparrow\rangle, |\downarrow\rangle$ that has $m_l + m_s = l + \frac{1}{2}$!

↓ lower

$$|j = l + \tfrac{1}{2},\ m_j = l - \tfrac{1}{2}\rangle \propto \hat{J}_-\left(|l, m_l = l\rangle |\uparrow\rangle\right)$$

$$= (\hat{L}_- + \hat{S}_-)|l, l\rangle|\uparrow\rangle$$

Skip!

Use: $\hat{L}_-|l, m_l\rangle = \hbar\sqrt{l(l+1) - m_l(m_l-1)}\,|l, m_l - 1\rangle$ and same for $s = \frac{1}{2}$:

$$\hat{S}_-|\uparrow\rangle = \hbar|\downarrow\rangle, \quad \hat{S}_-|\downarrow\rangle = 0$$

$$\Rightarrow |j = l + \tfrac{1}{2},\ m_j = l - \tfrac{1}{2}\rangle \propto \hbar\sqrt{\underbrace{l(l+1) - l(l-1)}_{2l}}\,|l, l-1\rangle + \hbar|l, l\rangle|\downarrow\rangle$$

$$\boxed{|j = l + \tfrac{1}{2},\ l - \tfrac{1}{2}\rangle \propto \sqrt{2l}\,|l, l-1\rangle + |l, l\rangle|\downarrow\rangle}$$, more generally:

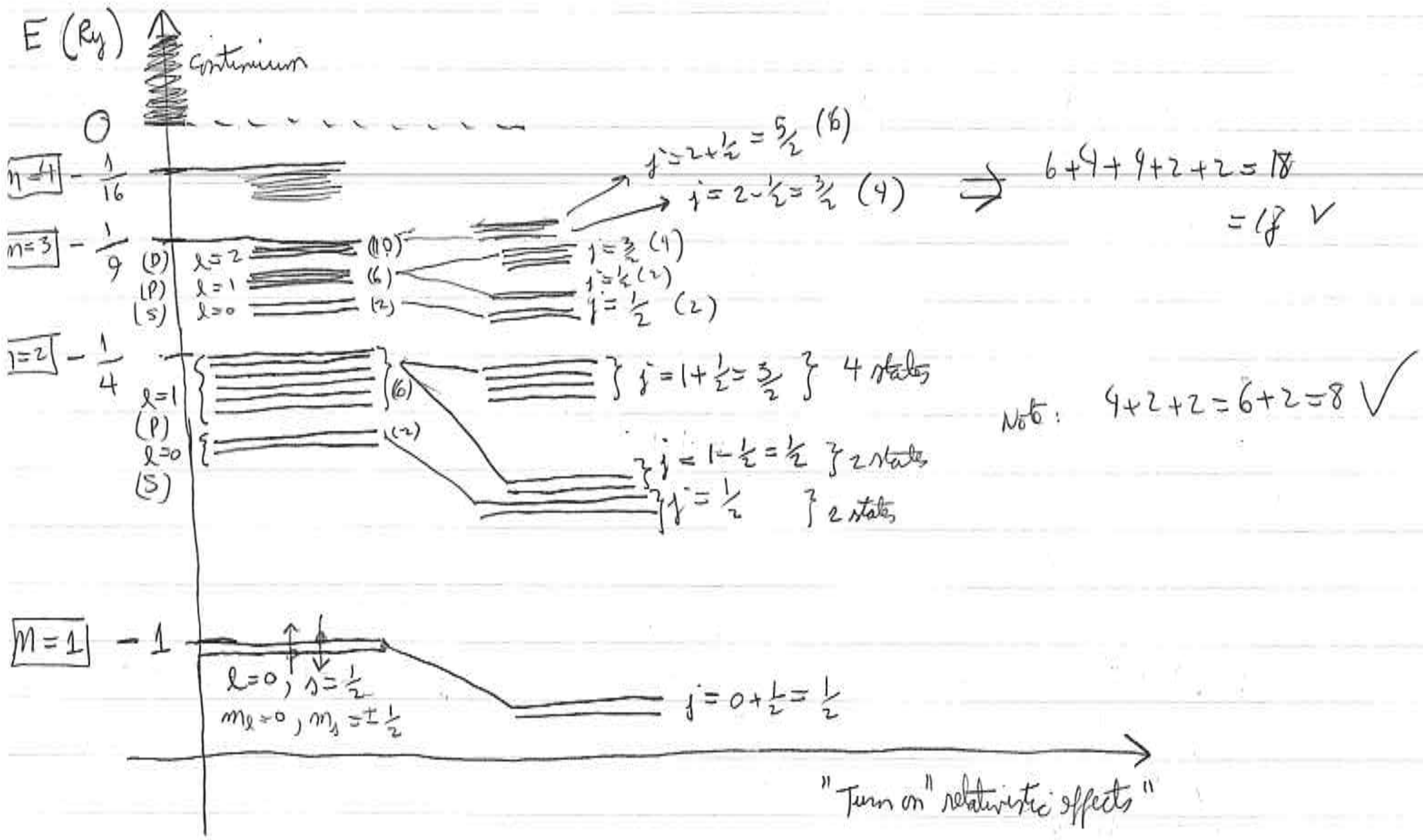
$$\Rightarrow |j\, m_j\rangle = \sum_{m_l + m_s = m_j} C^{l s, j}_{m_l, m_s, m_j} |l, m_l\rangle |s, m_s\rangle$$

$\longrightarrow$ Clebsh-Gordan coefficients.

~~We will get back to this shortly~~.

Show this after teaching addition of ang mom.!

Fine structure of Hydrogen: $E_{nj} = -\frac{Ry}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right]$



Addition of angular momenta [Griffiths section 4.4.3]

$$\underbrace{\vec{J}}_{2j+1} = \underbrace{\vec{J}_1}_{\substack{\text{size of}\\\text{Hilbert}\\\text{space}\\2j_1+1}} + \underbrace{\vec{J}_2}_{2j_2+1} \qquad \sum_{\text{allowed } j's} 2j+1 = 2j_1+1 + 2j_2+1$$

(e.g. $j_1 = j_2 = \frac{1}{2} \Rightarrow 2j_1+1 + 2j_2+1 = 4$ $\quad 2j+1 = 3 \quad j=1$; $\quad 2j+1 = 1 \quad j=0$)

Example: Consider $j_1 = j_2 = \frac{1}{2}$. Hilbert space is $\{|\uparrow\rangle, |\downarrow\rangle\} \otimes \{|\uparrow\rangle, |\downarrow\rangle\} =$

$$= \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}.$$

Note: $\vec{J}_1 = \vec{J}_1 \otimes \mathbb{1}$ in that $\hat{J}_{1z} |\uparrow\downarrow\rangle = \hat{J}_{1z} \otimes \mathbb{1} |\uparrow\downarrow\rangle = \hat{J}_{1z}|\uparrow\rangle \otimes \mathbb{1}|\downarrow\rangle$

$= +\frac{1}{2}\hbar |\uparrow\downarrow\rangle$)

and $\hat{J}_{2z} |\uparrow\downarrow\rangle = -\frac{1}{2}\hbar |\uparrow\downarrow\rangle$.

Also, $\vec{J}_2 = \mathbb{1} \otimes \vec{J}_2$.

Take a state like $|\uparrow\uparrow\rangle$: $\left(\hat{J}_{1z} + \hat{J}_{2z}\right)|\uparrow\uparrow\rangle = \underbrace{\left(\frac{1}{2} + \frac{1}{2}\right)}_{=1} \hbar |\uparrow\uparrow\rangle$

Take a state like $\alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle$: $\left(\hat{J}_{1z} + \hat{J}_{2z}\right)\left(\alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle\right) = 0$!

Propose:

$$\begin{cases} |\overset{j}{\downarrow}\!\!\!\!1, \overset{m_j}{\downarrow}\!\!\!\!1\rangle = |\uparrow\uparrow\rangle \\ |1, 0\rangle = \alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle \\ |1, -1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

Eigenstates of $(\hat{J}_{1z} + \hat{J}_{2z})$ are $1, 0, -1$

$$\begin{cases} |\uparrow\uparrow\rangle & m = 1 \\ |\uparrow\downarrow\rangle & m = 0 \\ |\downarrow\uparrow\rangle & m = 0 \\ |\downarrow\downarrow\rangle & m = -1 \end{cases}$$

$$\Big\{ |0, 0\rangle = \alpha'|\uparrow\downarrow\rangle + \beta'|\downarrow\uparrow\rangle$$

How to find α, β, α', β' ?

Use: $\hat{J}_-|1,1\rangle = \left(\hat{J}_{1-} + \hat{J}_{2-}\right)|\uparrow\uparrow\rangle$

$\hbar\underbrace{\sqrt{1\times(1+1) - 1\times(1-1)}}_{=2}\,|1,0\rangle = \hbar|\downarrow\uparrow\rangle + \hbar|\uparrow\downarrow\rangle \Rightarrow \boxed{|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)}$

$\Rightarrow \alpha = \beta = \frac{1}{\sqrt{2}}$!

α', β' ? $|0,0\rangle$ needs to be orthogonal to $|1,0\rangle$: $\langle 1,0|0,0\rangle = \frac{\alpha' + \beta'}{\sqrt{2}} = 0 \Rightarrow \boxed{\alpha' = -\beta'}$

Also, $\alpha'^2 + \beta'^2 = 1 \Rightarrow 2(\alpha')^2 = 1 \Rightarrow \alpha' = \frac{1}{\sqrt{2}}$, $\beta' = -\frac{1}{\sqrt{2}}$.

$\Rightarrow \boxed{|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}$.

$$\begin{cases} |1,1\rangle = |\uparrow\uparrow\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \qquad j=1 \text{ "triplet"} \\ |1,-1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

$$\left\{ |0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \qquad j=0 \text{ "singlet"} \right.$$

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0$$

Dimension of Hilbert space:

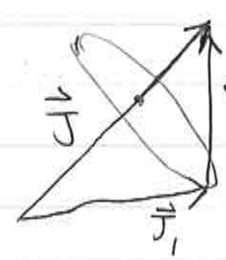
$$2 \times 2 = 2(1)+1 + 2(0)+1 = 3+1 = 4$$

In general: $\vec{J} = \vec{J}_1 + \vec{J}_2$ with $j_1 \geq j_2$:

$$j = (j_1 + j_2),\ j_1+j_2-1,\ j_1+j_2-2,\ \dots,\ (j_1 - j_2).$$

$$|j\ m\rangle = \sum_{m_1+m_2=m} \underbrace{C^{j_1 j_2 j}_{m_1 m_2 m}}_{\text{Clebsch-Gordan Coefficients}} |j_1 m_1\rangle |j_2 m_2\rangle$$

Physical picture:



For $(\vec{J})^2$ eigenstate, $\vec{J}_1, \vec{J}_2$ are not "constants of the motion", they precess.

See Griffiths page 188:

$1/2 \times 1/2$

m_1	m_2	$j=1,\ m=+1$
+1/2	+1/2	1

m_1	m_2	$j=1,\ m=0$	$j=0,\ m=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$j=1,\ m=-1$
−1/2	−1/2	1

$1 \times 1/2$

m_1	m_2	$j=3/2,\ m=+3/2$
+1	+1/2	1

m_1	m_2	$j=3/2,\ m=+1/2$	$j=1/2,\ m=+1/2$
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	$j=3/2,\ m=-1/2$	$j=1/2,\ m=-1/2$
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	$j=3/2,\ m=-3/2$
−1	−1/2	1

2×1

m_1	m_2	$j=3,\ m=+3$
+2	+1	1

m_1	m_2	$j=3,\ m=+2$	$j=2,\ m=+2$
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	$j=3,\ m=+1$	$j=2,\ m=+1$	$j=1,\ m=+1$
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

m_1	m_2	$j=3,\ m=0$	$j=2,\ m=0$	$j=1,\ m=0$
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	$j=3,\ m=-1$	$j=2,\ m=-1$	$j=1,\ m=-1$
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	$j=3,\ m=-2$	$j=2,\ m=-2$
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	$j=3,\ m=-3$
−2	−1	1

1×1

m_1	m_2	$j=2,\ m=+2$
+1	+1	1

m_1	m_2	$j=2,\ m=+1$	$j=1,\ m=+1$
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	$j=2,\ m=0$	$j=1,\ m=0$	$j=0,\ m=0$
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	$j=2,\ m=-1$	$j=1,\ m=-1$
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	$j=2,\ m=-2$
−1	−1	1

$2 \times 1/2$

m_1	m_2	$j=5/2,\ m=+5/2$
+2	1/2	1

m_1	m_2	$j=5/2,\ m=3/2$	$j=3/2,\ m=+3/2$
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	$j=5/2,\ m=+1/2$	$j=3/2,\ m=+1/2$
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	$j=5/2,\ m=-1/2$	$j=3/2,\ m=-1/2$
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	$j=5/2,\ m=-3/2$	$j=3/2,\ m=-3/2$
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	$j=5/2,\ m=-5/2$
−2	−1/2	1

$3/2 \times 1/2$

m_1	m_2	$j=2,\ m=+2$
+3/2	+1/2	1

m_1	m_2	$j=2,\ m=+1$	$j=1,\ m=+1$
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	$j=2,\ m=0$	$j=1,\ m=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$j=2,\ m=-1$	$j=1,\ m=-1$
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	$j=2,\ m=-2$
−3/2	−1/2	1

$3/2 \times 1$

m_1	m_2	$j=5/2,\ m=+5/2$
+3/2	+1	1

m_1	m_2	$j=5/2,\ m=+3/2$	$j=3/2,\ m=+3/2$
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	$j=5/2,\ m=+1/2$	$j=3/2,\ m=+1/2$	$j=1/2,\ m=+1/2$
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	$j=5/2,\ m=-1/2$	$j=3/2,\ m=-1/2$	$j=1/2,\ m=-1/2$
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	$j=5/2,\ m=-3/2$	$j=3/2,\ m=-3/2$
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	$j=5/2,\ m=-5/2$
−3/2	−1	1

Max j picture:

$|j_{max}, j_{max}\rangle =$
$= |j_1 j_1\rangle |j_2 j_2\rangle$
(No "precession")

Examples:

2 × 1 table, shaded column:

$$|3,0\rangle = \frac{1}{\sqrt{5}}|2,1\rangle|1,-1\rangle + \sqrt{\frac{3}{5}}\,|2,0\rangle|1,0\rangle + \frac{1}{\sqrt{5}}|2,-1\rangle|1,+1\rangle$$

3/2 × 1 table, shaded row:

$$\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle|1,0\rangle = \sqrt{\frac{3}{5}}\left|\tfrac{5}{2},\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{15}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$