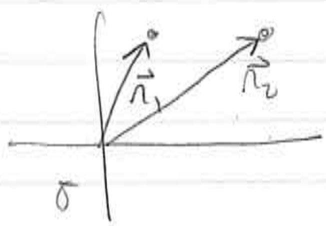
P423 - Lecture 10: Identical particles

Two particle systems

wave fnc depends on $\vec{r}_1$ and $\vec{r}_2$:



$$\Psi(\vec{r}_1, \vec{r}_2, t)$$

It satisfies a Schrödinger eqn just like one particle does,

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi,$$

$$\hat{H} = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2, t)$$

The interpretation is the same: For example,

$$|\Psi(\vec{r}_1, \vec{r}_2, t)|^2 d^3r_1\, d^3r_2$$

is the probability of finding particle 1 within d^3r_1 of $\vec{r}_1$ and particle 2 within d^3r_2 of $\vec{r}_2$. And

$$d^3r_2 \int d^3r_1 |\Psi(\vec{r}_1, \vec{r}_2, t)|^2$$

is the probability of finding particle 2 within d^3r_2 of $\vec{r}_2$ with particle 1 anywhere, etc...

②

So normalization

$$\int d^3r_1 \int d^3r_2 \, |\psi(\vec{r}_1, \vec{r}_2, t)|^2 = 1$$

Let's ignore spin for the moment. Suppose I made measurements and I found that particle 1 is in state $\psi_a(\vec{r})$ and particle 2 is in state $\psi_b(\vec{r})$. Then

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2)$$

It turns out that I made an implicit assumption: I assumed that I can tell the particles apart, i.e., that the particles are distinguishable (That they have "labels" 1 and 2 on them). In classical mechanics this is always the case. Just because in CM the position of each particle can be found with a measurement that in principle makes a very tiny disturbance on the particle. On the other hand, in QM it is not possible to detect the presence of a particle without causing a major disturbance (collapsing the wave fnc).

So if we have indistinguishable particles (such as two electrons) we need to construct a wave fnc that is non-committal as to which particle is in which state.

There are actually two ways of doing this: Symmetric (+) and antisymmetric (−):

$$\psi_\pm(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\,\psi_a(\vec{r}_2)\right]$$

It turns out that using relativistic QM we can show that there are two kinds of particles in nature:

- Integer spin $s = 0, 1, 2, 3, \ldots \Rightarrow$ BOSONS $\Rightarrow$ SYMMETRIC UNDER PARTICLE INTERCHANGE (+ sign).
- Half-integer spin $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \Rightarrow$ FERMIONS $\Rightarrow$ ANTISYMMETRIC UNDER PARTICLE INTERCHANGE (− sign).

This "Spin-statistics theorem" is a consequence of Lorentz invariance (under a boost to a new reference frame moving w.r.t. the old ref. frame).

This result has a drastic consequence in the structure of matter.

For example, two fermions can not occupy the same quantum state: Try to form the wave fn:

$$\psi_-(\vec{r}_1,\vec{r}_2) = A\left[\psi_a(\vec{r}_1)\,\psi_a(\vec{r}_2) - \psi_a(\vec{r}_1)\,\psi_a(\vec{r}_2)\right] = 0$$

You can't write a wave fn that is antisymmetric with two particles occupying the same state!

This is the Pauli exclusion principle.

Formalization:

More formally, define the particle interchange operator

$$\hat{P}_{12}\,\psi(\vec{r}_1,\vec{r}_2) = \psi(\vec{r}_2,\vec{r}_1)$$

Because $\hat{P}_{12}^2 = 1$ (check!) $\hat{P}_{12}$ has two eigenvalues: Either $+1$ or -1. If the particles are indistinguishable we will have (For example, $m_1 = m_2$ and $V(\vec{r}_1,\vec{r}_2) = V(\vec{r}_2,\vec{r}_1)$):

$$\hat{H}(\vec{p}_1,\vec{p}_2,\vec{r}_1,\vec{r}_2) = H(\vec{p}_2,\vec{p}_1,\vec{r}_2,\vec{r}_1) = P_{12}^\dagger H P_{12}$$

$$\Rightarrow H = P_{12}^\dagger H P_{12} \Rightarrow P_{12}H - HP_{12} = 0 \Rightarrow \boxed{[P_{12}, H] = 0}.$$

If a particle starts in a state

$$\hat{P}_{12}\,\psi_+(\vec{r}_1,\vec{r}_2) = \psi_+(\vec{r}_2,\vec{r}_1) = +\,\psi_+(\vec{r}_1,\vec{r}_2) \quad \text{(symmetric)}$$

or $$\hat{P}_{12}\,\psi_-(\vec{r}_1,\vec{r}_2) = -\,\psi_-(\vec{r}_1,\vec{r}_2). \quad \text{(antisymmetric)}.$$

it will remain in that state. Proof:

$$|\psi(t=0)\rangle = |\psi_+(0)\rangle \Rightarrow |\psi(t)\rangle = e^{-i\frac{Ht}{\hbar}}|\psi_+(0)\rangle$$

$$\hat{P}_{12}|\psi(t)\rangle = e^{-i\frac{Ht}{\hbar}}\hat{P}_{12}|\psi_+(0)\rangle = +\,e^{-i\frac{Ht}{\hbar}}|\psi_+(0)\rangle = +|\psi(t)\rangle$$

$$[\hat{P}_{12}, H] = 0 \qquad\qquad \Rightarrow |\psi(t)\rangle = |\psi_+(t)\rangle.$$

In reality, all particles are required to be either in + or − states always. This is a new postulate, the symmetrization requirement.

Example: Find the energies of two non-interacting particles that are bound in the infinite square well, when:

a) They are distinguishable

b) They are Bosons

c) They are Fermions

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m}\frac{d^2}{dx_2^2} + V(x_1) + V(x_2) \qquad V(x_i) = \begin{cases} 0, & 0 < x < a \\ \infty & \text{otherwise} \end{cases}$$

$$\hat{H}\psi(\vec{r}_1, \vec{r}_2) = E\,\psi(\vec{r}_1, \vec{r}_2)$$

Diff. eqn is separable, so

$$\psi(x_1, x_2) = \psi_{n_1}(x_1)\,\psi_{n_2}(x_2)$$

are solutions for the case (a). Here

$$\psi_{n_i}(\vec{r}) = \sqrt{\frac{2}{a}}\sin\left(\frac{n_i\pi}{a}x\right), \quad E_{n_i} = \underbrace{\frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2}_{\equiv K} n_i^2 = K n_i^2$$

So for case (a), the allowed energies are

$$E_{n_1 n_2} = K(n_1^2 + n_2^2) \qquad \begin{array}{l} n_1 = 1, 2, 3, \ldots \\ n_2 = 1, 2, 3, \ldots \end{array} \quad \Rightarrow \frac{E}{K} = 2, 5, 8, 10, \ldots$$

degeneracies: 1, 2, 1, 2

e.g. ground state has $E = 2K$ and 1st excited state $E = 5K$ is doubly degenerate: Either $n_1 = 1$ and $n_2 = 2$, or $n_1 = 2$ and $n_2 = 1$.

b) Bosons: $\psi(x_1, x_2) = A\left[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)\right]$

So $n_1 = n_2$ is allowed. So the ground state is the same:

$$\psi_{\substack{\text{ground}\\\text{Boson}}}(x_1, x_2) = \psi_1(x_1)\psi_1(x_2)$$ but the 1st excited state can only be:

$$\psi_{\substack{E=5K\\\text{Bosons}}} = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right].$$

So the $E = 5K$ state is non-degenerate for Bosons! So $\frac{E}{K} = 2, 5, 8, 10, \ldots$ all non-degenerate!

c) Fermions: The ground state can not be $n_1=1, n_2=1$. It is actually

$$\psi_{\substack{ground\\ Fermion}} = \left[\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)\right]$$

and it has $E = 5k$! Only two particle states with $n_1 \neq n_2$ are allowed, i.e.

$$E_{n_1 n_2} = k\left[n_1^2 + n_2^2\right] \quad \text{with } n_1 \neq n_2, \text{ so}$$

$$\frac{E}{k} = 5, 10, 13, 17, 20, \ldots \quad \text{all non-degenerate.}$$

Note: The allowed Fermion energies are a subset of the Boson energies!