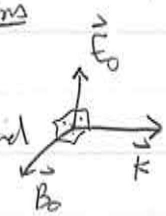
P423 - Lecture 18: Emission and absorption of radiation

Consider an atom that is subject to electromagnetic radiation, AKA light:

Maxwell's eqns

$$\begin{cases} \vec{E} = \vec{E}_0 \cos(kx - \omega t) \\ \vec{B} = \vec{B}_0 \cos(kx - \omega t) \end{cases}$$

with $\omega = ck$, $|\vec{E}_0| = c|\vec{B}_0|$ and



If the size of the atom is much less than $\lambda = \frac{2\pi}{k}$ then we can approximate

$$\begin{cases} \vec{E} \approx \vec{E}_0 \cos(\omega t) \\ \vec{B} \approx \vec{B}_0 \cos(\omega t) \end{cases}$$

Assuming no magnetic field is applied, $\omega > 0$ will be resonant only with the electric field.

So we can assume the atom is subject to $\vec{E}$ field only, described by a potential:

$$H' = -q\,\vec{E} \cdot \vec{r} \qquad \left(\text{Note } q\vec{E} = -\vec{\nabla} H'\right)$$

Let's use our time-dependent pert. theory to calculate transition rates: Assuming ψ_b is an excited state and ψ_a is a ground state:

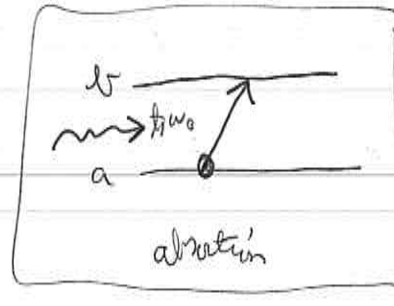
$$H'_{ba} = \langle \psi_b | (-q\vec{E} \cdot \vec{r}) | \psi_a \rangle = -\underbrace{q\langle \psi_b | \vec{r} | \psi_a \rangle}_{\vec{P}_{ba},\ \text{electric dipole moment}} \cdot \vec{E}_0 \cos(\omega t)$$

Let's assume $\vec{E}_0 = E_0 \hat{n}$ so that

$$H'_{ba} = \underbrace{-(\vec{P}_{ba} \cdot \hat{n})\, E_0}_{V_{ba}} \cos(\omega t)$$

According to our theory:

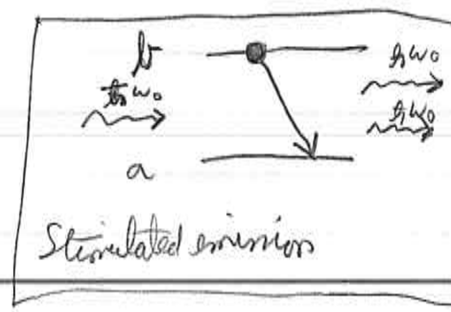
$$P_{a\to b} = \left[\frac{(\vec{P}_{ba}\cdot\hat{n})E_0}{\hbar}\right]^2 \frac{\sin^2\left[(\omega_0-\omega)\frac{t}{2}\right]}{(\omega_0-\omega)^2}$$



This gives the prob. for the atom to absorb a photon with energy $\hbar\omega_0$ from the EM field $(E_b - E_a = \hbar\omega_0)$.

But how about $P_{b\to a} = |c_a(t)|^2$ when $c_b(0)=1$ and $c_a(0)=0$? We can obtain through the same procedure, by $a \rightleftarrows b$ and $\omega_0 \to -\omega_0$, but we must pick the other denominator in $\frac{e^{i(\omega_0+\omega)t}-1}{(\omega_0+\omega)} + \frac{e^{i(\omega_0-\omega)t}-1}{(\omega_0-\omega)}$:

$$P_{b\to a} = \left[\frac{(\vec{P}_{ab}\cdot\hat{n})E_0}{\hbar}\right]^2 \frac{\sin^2\left[(\omega_0-\omega)\frac{t}{2}\right]}{(\omega_0-\omega)^2}$$

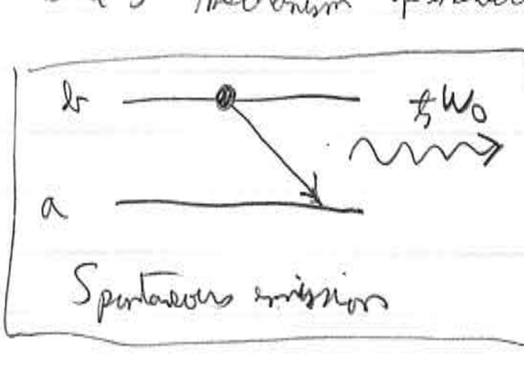


In this case EM field "gains" $\hbar\omega_0$ of energy.

$\Rightarrow$ This leads to amplification.

Principle of the laser: Create a "~~populat~~ inversion" of b levels and shine some light on it. By chain reaction the end result is a cascade of photons with same frequency and phase.
(Light Amplification by Stimulated Emission of Radiation).

It turns out that there is a 3rd mechanism of interaction of light with matter that is called spontaneous emission:



we will show that such a process must exist otherwise matter would never attain thermal equilibrium.

In a way spontaneous emission is the result of "stimulated emission from the zero point fluctuations of the EM field. Even with zero photons, the EM field has energy $\sum_{\omega_0} \frac{\hbar\omega_0}{2}$. This zero point vibration stimulates the decay of atoms from an excited state to their ground state.

Before we calculate the rate for this process, let us compute the rate for absorption and stimulated emission for an atom in a gas of photons at thermal equilibrium. These photons are incoherent in the sense that they are distributed according to $\rho(\omega)$ (Blackbody radiation density) and the direction $\hat{n}$ of their $\vec{E}$ field is averaged over the sphere.

The energy density of an EM wave is:

$$\mu = \frac{\varepsilon_0}{2}\langle(\vec{E})^2\rangle + \frac{1}{2\mu_0}\langle(\vec{B})^2\rangle = \frac{\varepsilon_0}{2}E_0^2\underbrace{\langle\cos^2(\omega t)\rangle}_{=\frac{1}{2}} + \frac{1}{2\mu_0}\underbrace{B_0^2}_{=\frac{E_0^2}{c^2}}\underbrace{\langle\cos^2(\omega t)\rangle}_{=\frac{1}{2}}$$

$$= \frac{\varepsilon_0 E_0^2}{4} + \frac{E_0^2}{4\mu_0 c^2} = \frac{1}{2}\varepsilon_0 E_0^2 .$$

$$P_{b\to a} = \overset{E_0^2}{\frac{2\mu}{\varepsilon_0\hbar^2}}\left(\vec{P}_{ba}\cdot\hat{n}\right)^2 \frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$$

In a blackbody, $\mu \to \rho(\omega)\,d\omega$ and we integrate over all ω with

$$(\vec{P}_{ba}\cdot\hat{n})^2 \to \langle(\vec{P}_{ba}\cdot\hat{n})^2\rangle$$

④

$$P_{b\to a}(t) = \frac{2}{\varepsilon_0 \hbar^2}\langle(\vec{p}_{ba}\cdot\hat{n})^2\rangle \int_0^\infty d\omega \underbrace{\rho(\omega)}_{\text{Broad}} \underbrace{\left[\frac{\sin^2\left(\frac{(\omega_0-\omega)t}{2}\right)}{(\omega_0-\omega)^2}\right]}_{\text{Sharply peaked at } \omega=\omega_0}$$

$$\approx \frac{2}{\varepsilon_0\hbar^2}\langle(\vec{p}_{ba}\cdot\hat{n})^2\rangle \rho(\omega_0)\int_0^\infty d\omega \frac{\sin^2\left[\frac{(\omega_0-\omega)t}{2}\right]}{(\omega_0-\omega)^2}$$

use $x = \frac{(\omega_0-\omega)t}{2}$ and get

$$-\frac{2}{t}\frac{t^2}{4}\int_{\frac{\omega_0 t}{2}}^{-\infty} dx \frac{\sin^2(x)}{x^2} = \frac{t}{2}\int_{-\infty}^{\frac{\omega_0 t}{2}} dx \frac{\sin^2(x)}{x^2} \approx \frac{t}{2}\underbrace{\int_{-\infty}^{\infty} dx \frac{\sin^2(x)}{x^2}}_{=\pi}$$

when $\omega_0 t >> 1$ we can extend this integral to $\int_{-\infty}^{\infty}$

$$\Rightarrow P_{b\to a}(t) \approx \frac{\pi}{\varepsilon_0\hbar^2}\langle(\vec{p}_{ba}\cdot\hat{n})^2\rangle\rho(\omega_0)\, t$$

Calculate $\langle(\vec{p}_{ba}\cdot\hat{n})^2\rangle$ (assume $\vec{p}_{ba} \parallel \hat{z}$)

$$= \frac{|\vec{p}_{ba}|^2}{4\pi}\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin(\theta)\cos^2(\theta) = \frac{|\vec{p}_{ba}|^2 2\pi}{4\pi}\int_{-1}^{1} d(\cos\theta)\cos^2(\theta)$$

$$= \frac{|\vec{p}_{ba}|^2}{2}\left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{|\vec{p}_{ba}|^2}{2}\frac{2}{3} = \frac{|\vec{p}_{ba}|^2}{3}$$

$$\boxed{P_{b\to a}(t) = \frac{\pi|\vec{p}_{ba}|^2}{3\varepsilon_0\hbar^2}\rho(\omega_0)\, t} = \Gamma_{b\to a}\, t$$

It is convenient to think of a rate (per unit time) $\Gamma_{b\to a}$.

Another way to derive this result, Fermi's golden rule :

$$H' = -\frac{(\vec{p}\cdot\hat{n})}{2}\mathcal{E}_0 e^{i\omega t} - \frac{(\vec{p}\cdot\hat{n})}{2}\mathcal{E}_0 e^{-i\omega t} = \hat{V}' e^{i\omega t} + (\hat{V}')^{\dagger} e^{-i\omega t} \quad \text{where } \hat{V}' = -\frac{\hat{\vec{p}}\cdot\hat{n}}{2}\mathcal{E}_0 = \frac{\hat{V}}{2}$$

$$P_{b\to a} = \frac{4|V'_{ba}|^2}{\hbar^2} \underbrace{\frac{\sin^2\left[(\omega_0-\omega)\frac{t}{2}\right]}{(\omega_0-\omega)^2}} = \Gamma_{b\to a}\, t$$

when $t\to\infty$, $(t \gg \frac{2}{\omega_0})$ assume $\approx \frac{\pi}{2} t\, \delta(\omega-\omega_0) = \frac{\pi t}{2}\hbar\, \delta[\hbar\omega - (E_b - E_a)]$

$$\boxed{\Gamma_{b\to a}(\omega) \approx \frac{2\pi}{\hbar}|V'_{ba}|^2 \delta\underbrace{\left[\hbar\omega - (E_b - E_a)\right]}_{\text{ensures conservation of energy}}}$$

"Fermi's Golden rule"

To get rate for a Blackbody,

$$\Gamma^{Total}_{b\to a} = \sum_\omega \Gamma(\omega) = \sum_\omega \frac{2\pi}{\hbar}\frac{\langle(\vec{p}\cdot\hat{n})^2\rangle \underbrace{E_0^2}_{\frac{2u}{\varepsilon_0}}}{4}\delta(\hbar\omega - \hbar\omega_0)$$

$$= \sum_u u\, \frac{4\pi}{4\hbar^2\varepsilon_0}\langle(\vec{p}\cdot\hat{n})^2\rangle\, \delta(\omega-\omega_0)$$

$$= \frac{\pi}{\hbar^2\varepsilon_0}\langle(\vec{p}\cdot\hat{n})^2\rangle \int d\omega\, \rho(\omega)\, \delta(\omega-\omega_0)$$

$$\boxed{\Gamma^{Total}_{b\to a} = \frac{\pi}{\hbar^2\varepsilon_0}\underbrace{\langle(\vec{p}_{ba}\cdot\hat{n})^2\rangle}_{=|\vec{p}_{ba}|^2/3}\rho(\omega_0)}$$