University of Victoria

Final examination

December 12th, 2014

QUANTUM MECHANICS II (PHYS 423)

Professor: R. de Sousa

Duration: 3 hours – Total credit: 60

NAME:________________________________

STUDENT NUMBER: V00________________________________

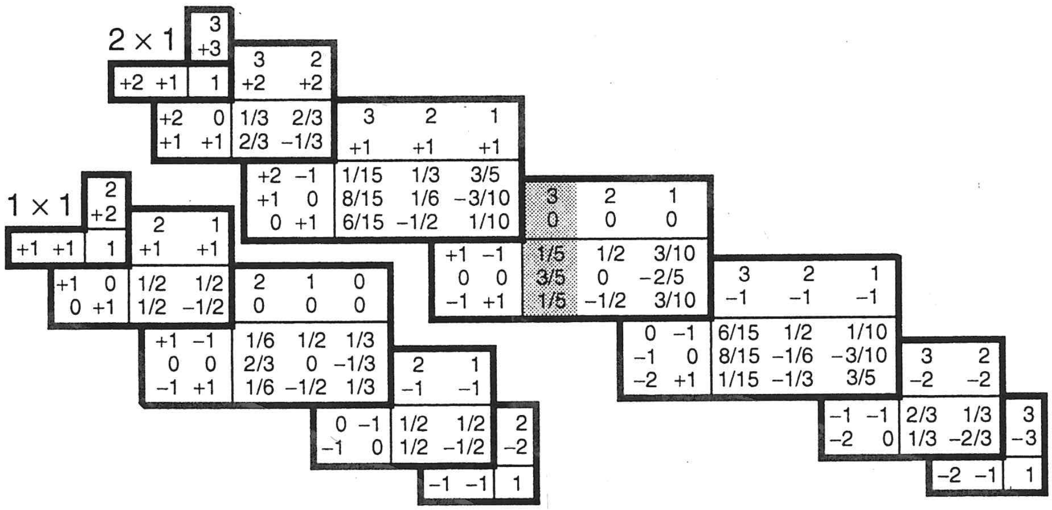
INSTRUCTIONS:

- Write your answers into the space provided for each problem. Clearly explain your reasoning. If you need more space, please use the back of the page.
- This exam has a total of 7 pages including this cover page; there are 6 problems.
- Students must count the number of pages and report any discrepancy to the invigilator.
- This examination must be answered on the question paper.
- Items permitted: Students are allowed one formula sheet, handwritten on both sides of an 8.5×11 inch page, and one calculator model Sharp EL-510R. No other items such as books, computers, or cell phones are allowed.
- Write your name and student number in the space provided at the top of this page.

1. **(/10) Addition of angular momenta.–** $\boldsymbol{S}_1$ is a spin-1 angular momentum operator, and $\boldsymbol{S}_2$ is a spin-2 angular momentum operator.

   (a) What are the eigenvalues of the operator $(\boldsymbol{S}_1 + \boldsymbol{S}_2)^2$?

   (b) For each different eigenvalue, write down *one eigenvector* explicitly as a linear combination of $|2, m\rangle|1, m'\rangle$.



2. **(/10) Eigenstates and eigenenergies for two identical particles in a two level system.–** The single particle Hamiltonian $\mathcal{H}$ can assume one of the two orbital states: State $\psi_a(\boldsymbol{r})$ with energy E_a and state $\psi_b(\boldsymbol{r})$ with energy E_b ($\langle\psi_i|\psi_j\rangle = \delta_{ij}$). Assuming that $\mathcal{H}$ is independent of spin, write down *all* allowed two-particle energy eigenstates and their corresponding eigenenergies for the cases below. Make sure you normalize your wave functions and show the coordinates $\boldsymbol{r}_1, \boldsymbol{r}_2$ and spin states explicitly.

   (a) The particles are spin-0 Bosons.

   (b) The particles are spin-1/2 Fermions.

3. **(/10) Particle in a 1d triangular well.**– A particle in one dimension is subject to the following confining potential:

$$V(x) = \begin{cases} \infty & \text{for } x \leq 0, \\ c\,x & \text{for } x > 0, \end{cases}$$

where c is a positive constant. Using the variational state

$$\psi_0(x) = Ax\mathrm{e}^{-\alpha x}$$

for $x \geq 0$, where A is a normalization constant and α is the variational parameter, find the approximate ground state energy.
Useful integral:

$$\int_0^\infty x^n \mathrm{e}^{-x/a} dx = n!\; a^{n+1}. \tag{1}$$

4. **(/10) Perturbation theory applied to the 3d harmonic oscillator.**– Consider the isotropic three-dimensional harmonic oscillator subject to the perturbation

$$\mathcal{H}' = \lambda x^2 yz. \tag{2}$$

Calculate the energy level shifts *that are linear in* λ for the following cases:

(a) The ground state;

(b) The (triply degenerate) first excited state.

Hint: You can use either creation/destruction operators, or calculate the matrix elements explicitly using integrals of the wave functions. For the latter method, you may need the wave functions for the ground and 1st excited state of the **one dimensional** *oscillator:*

$$\psi_0(x) = \frac{1}{\pi^{1/4} x_0^{1/2}} \mathrm{e}^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}, \quad \psi_1(x) = \frac{2^{1/2}}{\pi^{1/4} x_0^{1/2}} x \mathrm{e}^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2},$$

where $x_0 = \sqrt{\hbar/(m\omega_0)}$, and ω_0 is the frequency of the oscillator. You may also need the integral $\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-(x/a)^2} dx = \pi^{1/2} a^3/2$.

5. **(/10) Spontaneous emission for the particle in a box.**– Consider a one dimensional particle in a box, subject to the potential

$$V(x) = \begin{cases} 0 & \text{for } -\frac{a}{2} < x < \frac{a}{2}, \\ \infty & \text{for } x \leq -\frac{a}{2} \text{ and } x \geq \frac{a}{2}. \end{cases}$$

The particle is assumed to have charge q.

(a) Find the energy eigenstates and eigenvalues.

(b) Find the selection rule(s) for spontaneous emission of electromagnetic radiation (no need to calculate the rates explicitly).

(c) Write down an expression for the spontaneous emission *lifetime* of a particle in the state $\psi = \sqrt{2/a}\sin(4\pi x/a)$ in terms of electric dipole matrix elements (again, no need to calculate the matrix elements explicitly).

6. **(/10) Phase shift for a 1d scattering problem.–** A particle of mass m and energy E is incident from the left on the potential

$$V(x) = \begin{cases} 0 & \text{for } x < -a, \\ -V_0 & \text{for } -a \leq x < 0, \\ \infty & \text{for } x \geq 0. \end{cases}$$

   (a) If the incoming wave is Ae^{ikx} (where $k = \sqrt{2mE}/\hbar$), and the reflected wave is Be^{-ikx}, find the reflected wave amplitude B.

   (b) Using your result above, show that the reflected wave has the same amplitude as the incident wave, as expected from conservation of probability.

   (c) Define the phase shift δ as $B = -Ae^{2\delta i}$. Calculate δ for a very deep well satisfying $V_0 \gg E$.