An Introduction to Spectral Graph Theory

Mackenzie Wheeler
Supervisor: Dr. Gary MacGillivray
University of Victoria
WheelerM@uvic.ca
Outline

1. How many walks are there from vertices $v_i$ to $v_j$ of length $d$?
2. How many labelled spanning trees of $G$ exist?
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1. **How many walks are there from vertices** $v_i$ **to** $v_j$ **of length** $d$?

2. **How many labelled spanning trees of** $G$ **exist?**

   - Graph Theory Review
   - Define the Adjacency matrix $A(G)$
   - Answer Question 1
   - Linear Algebra Review
   - Define the Laplacian matrix $L(G)$
   - Answer Question 2
Graph Theory Review

Definition
Two vertices $v_i$ and $v_j \in V(G)$ are said to be adjacent if
$
\{v_i, v_j\} \in E(G).
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Graph Theory Review

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Definition
A walk in a graph $G$ is a sequence of vertices \{v_1, v_2, \ldots, v_k\} such that $v_i$ is adjacent to $v_{i+1}$ for all $1 \leq i \leq k - 1$. The length of the walk is $k - 1$. 
The Adjacency Matrix

Definition
The adjacency matrix $A(G)$ of a graph $G$ is defined by

$$(A(G))_{ij} = \begin{cases} 
1 & v_i v_j \in E(G) \\
0 & v_i v_j \notin E(G)
\end{cases}$$
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Example
Let $G = C_5$, then we have that

$$A(C_5) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$
Counting walks

Question

Given a graph $G$, how many walks are there from $v_i$ to $v_j$ of length $d$?
Counting walks

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Proposition
Let $G$ be a graph with $n$ vertices and adjacency matrix $A(G)$, then the number of walks from $v_i$ to $v_j$ of length $d$ in $G$ is given by $(A(G))_{ij}^d$. 
The Adjacency Matrix

Proof.

$A_d$ is $A$.

Consider $A_{d+1} = A_d A$.

Then $a_{d+1}^{ij} = \sum_{n=1}^{N} a_d^{ik} a_{kj}$.

$a_d^{ik} a_{kj}$ is the number of walks from $v_i$ to $v_j$ which are walks from $v_i$ to $v_k$ of length $d$, followed by a walk of length 1 from $v_k$ to $v_j$.

Therefore $a_{d+1}^{ij} = \sum_{n=1}^{N} a_d^{ik} a_{kj}$ is the total number of walks from $v_i$ to $v_j$ of length $d+1$. 
The Adjacency Matrix

Proof.

- For \( d = 1 \), \( A^d \) is \( A \).
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- Consider $A^{d+1} = A^d A$
- Then $a_{ij}^{d+1} = \sum_{k=1}^{n} a_{ik}^d a_{kj}$.
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Proof.

- For \( d = 1 \), \( A^d \) is \( A \).
- Consider \( A^{d+1} = A^d A \)
- Then \( a_{ij}^{d+1} = \sum_{k=1}^{n} a_{ik}^d a_{kj} \).
- \( a_{ik}^d a_{kj} \) is the number of walks from \( v_i \) to \( v_j \) which are walks from \( v_i \) to \( v_k \) of length \( d \), followed by a walk of length 1 from \( v_k \) to \( v_j \).
- Therefore \( a_{ij}^{n+1} = \sum_{k=1}^{n} a_{ik}^d a_{kj} \) is to total number of walks from \( v_i \) to \( v_j \) of length \( d + 1 \).
The Adjacency Matrix

Corollary

Let $G$ be a graph with $e$ edges and $t$ triangles, then

1. $tr(A(G)^2) = 2e$

2. $tr(A(G)^3) = 6t$
The Laplacian Matrix

Definition

The Laplacian matrix $L(G)$ of a graph $G$ is defined by

$$(L(G))_{ij} = \begin{cases} 
\deg(v_i) & i = j \\
-1 & i \neq j \text{ and } v_i v_j \in E(G) \\
0 & \text{otherwise}
\end{cases}$$
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\end{cases}$$

Example
Let $G = C_5$, then we have that

$$L(C_5) = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 & 2 \end{bmatrix}$$
Definition
Let $A \in M_{n\times n}(\mathbb{R})$ and let $v \in \mathbb{R}^n$ be a nonzero vector. Then $v$ is an eigenvector of $A$ if there exists a scalar $\lambda \in \mathbb{R}$, such that $Av = \lambda v$. We say that $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $v$. 

Proposition
Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix, then the eigenvalues of $A$ are all real numbers.
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Proposition
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Linear Algebra Review

Definition
Let $A \in M_{n \times n}(\mathbb{R})$, and let $a_{ij}$ denote the entry in the $i^{th}$ row and $j^{th}$ column. $A$ is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

for all $1 \leq i \leq n$. 

Example
$$A = \begin{bmatrix}
7 & 1 & 0 & 2 \\
-1 & -1 & 6 & -1 \\
0 & -1 & 3 & 1 & 0 \\
0 & -1 & 2 & 0 & -1
\end{bmatrix}$$
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Example

$$A = \begin{bmatrix}
7 & 1 & 0 & 2 & -1 \\
-1 & 6 & -1 & 0 & 0 \\
0 & -1 & 3 & 1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
2 & 0 & 0 & -1 & 4
\end{bmatrix}$$
Proposition

Let $A$ be a symmetric, diagonally dominant $n \times n$ matrix such that $a_{ii} > 0$ for all $a \leq i \leq n$. Then all the eigenvalues of $A$ are non-negative.
Linear Algebra Review

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Let $A$ be a symmetric, diagonally dominant $n \times n$ matrix such that $a_{ii} > 0$ for all $a \leq i \leq n$. Then all the eigenvalues of $A$ are non-negative.

Corollary

Let $G$ be a graph with Laplacian $L(G)$. The eigenvalues of $L(G)$ are all nonnegative real numbers. Therefore, we may list the eigenvalues of $L(G)$ as $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. 
Proposition
Let $G$ be a graph with Laplacian matrix $L(G)$. Then $\lambda = 0$ is an eigenvalue of $L(G)$ with $v = (1, 1, \ldots, 1)$ as a corresponding eigenvector.

Proposition
Let $G$ be a connected graph with Laplacian $L(G)$. Then $\lambda = 0$ is an eigenvalue of $L(G)$ with multiplicity one.
The Laplacian Matrix

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Counting Labelled Spanning Trees

Definition
A graph \( G \) is a tree if \( G \) is connected and contains no cycles.

Definition
Let \( G \) be a graph with a subgraph \( T \). \( T \) is a spanning tree of \( G \) if \( V(T) = V(G) \) and \( T \) is a tree.
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The Petersen Graph
Counting Labelled Spanning Trees

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Let $G$ be a graph with a subgraph $T$. $T$ is a spanning tree of $G$ if $V(T) = V(G)$ and $T$ is a tree.

**Example**

A spanning tree of the Petersen graph
Counting Labelled Spanning Trees

Question

*Given a graph* $G$ *with vertices labelled* $\{v_1, v_2, \ldots, v_n\}$ *how many labelled spanning trees of* $G$ *exist?*
Counting Labelled Spanning Trees

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Given a graph $G$ with vertices labelled $\{v_1, v_2, \ldots, v_n\}$ how many labelled spanning trees of $G$ exist?

Theorem (Kirchoff’s Theorem)
Let $G$ be a connected graph with $n \geq 2$ labelled vertices, and let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$ be the eigenvalues of $L(G)$. Then the number of spanning trees on $G$, $t(G)$, is given by

$$t(G) = \det(L(G)[i]) = \frac{1}{n} \prod_{k=1}^{n-1} \lambda_k.$$

Where $L(G)[i]$ denotes the matrix obtained from $L(G)$ by deleteing the $i^{th}$ row and $i^{th}$ column.
Counting Labelled Spanning Trees

**Proof Outline:**

We proceed by induction on $|V(G)| + |E(G)| = n + m$.

When $n + m = 3$, the only connected graph is $P_2$. Thus, $L(P_2) = \begin{bmatrix} 1 \end{bmatrix}$. 
Counting Labelled Spanning Trees

**Proof Outline:**

- We proceed by induction on $|V(G)| + |E(G)| = n + m$

  When $n + m = 3$, the only connected graph is $P_2$.

  $L(P_2) = \begin{bmatrix} 1 - 1 & 1 \\ 1 & 1 - 1 \end{bmatrix}$
Counting Labelled Spanning Trees

Proof Outline:

- We proceed by induction on \(|V(G)| + |E(G)| = n + m|
- When \(n + m = 3\), the only connected graph is \(P_2\)
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$$P_2 \begin{array}{c}
  \bullet \\
  \big|
\end{array} \begin{array}{c}
  \bullet \\
  \big|
\end{array}$$

$$L(P_2) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
Counting Labelled Spanning Trees

Proof Outline:
Counting Labelled Spanning Trees

**Proof Outline:**

- Consider a graph $|V(G)| + |E(G)| = n + m + 1$
Counting Labelled Spanning Trees

Proof Outline:

- Consider a graph $|V(G)| + |E(G)| = n + m + 1$
- Let $e = v_i v_j$ be an edge incident with the vertex $v_i$
Counting Labelled Spanning Trees

Proof Outline:

- Consider a graph $|V(G)| + |E(G)| = n + m + 1$
- Let $e = v_i v_j$ be an edge incident with the vertex $v_i$
- Notice that $t(G) = t(G - e) + t(G \setminus e)$
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- Notice that $t(G) = t(G - e) + t(G \setminus e)$

![Diagram of a graph with vertices and edges](attachment:graph_diagram.png)

$L(G \setminus e) = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$
Proof Outline:

By standard manipulation of the determinant we get

\[
\det(L(G)[i]) = \det(L(G - e)[i]) + \det(L(G \setminus e)[j]) \\
= t(G - e) + t(G \setminus e), \quad \text{by induction hypothesis} \\
= t(G).
\]
Corollary (Cayley’s Formula)

The number of labelled spanning trees on the complete graph $K_n$ is $n^{n-2}$.
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The number of labelled spanning trees on the complete graph $K_n$ is $n^{n-2}$.

Proof.
The eigenvalues of $L(K_n)$ are 0 and $n$ with multiplicity 1 and $n - 1$, respectively. Therefore, by Kirchoff’s Theorem the number of spanning trees on $K_n$ is $\frac{n^{n-1}}{n} = n^{n-2}$. \qed
References
