MULTIPLICATIVE ERGODIC THEOREMS AND APPLICATIONS

ANTHONY QUAS

Lecture notes for a minicourse given at the Universidade de São Paulo

0. Outline, Key Notation and Formatting conventions

I cover in some detail the material on Multiplicative Ergodic Theorems. The applications part is treated only briefly here.

I want to record a couple of notational conventions that I will use consistently. Oseledets exponents will be listed in strictly *decreasing* order:

$$\lambda_1 > \lambda_2 > \ldots > \lambda_k \text{ with multiplicities } m_1, m_2, \ldots, m_k.$$

$V_j(\omega)$ will be the jth 'slow space', that is things expanding at rate λ_j or less.

$U_j(\omega)$ will be a space expanding exactly at rate λ_j.

The base dynamics will be denoted by a map σ from a probability space $(\Omega, \mathbb{P})$ to itself.

> *This is when I'm trying to say something 'unofficial'. These are meant to be friendly comments.*

> I'll use 'double boxes' for point(s) I really want to emphasize

1. Motivation: Multiplicative Ergodic Theorems

> *Only read this section if you're not already comfortable with the statement of the multiplicative ergodic theorem.*

If $\sigma\colon \Omega \to \Omega$ is a dynamical system, and we have a map $A\colon \Omega \to M_d(\mathbb{R})$, we study compositions $A_\omega^{(n)} = A_{\sigma^{n-1}\omega} \cdots A_\omega$.

The original motivation for this comes from differentiable dynamical systems. If Ω is a manifold, σ is a differentiable map from Ω to itself and

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A_ω is the derivative of σ at ω (in a suitable local chart), then the product $A_\omega^{(n)}$ is the derivative of σ^n. We can look for expanding/contracting directions etc.

If ω is a fixed point of σ, $A_\omega^{(n)}$ is just A_0^n, where A_0 is the derivative matrix at the fixed point. In this case, the Jordan normal form implies that A_0 is similar to an upper triangular matrix in Jordan form. In particular, there are eigenvalues $\alpha_1, \ldots, \alpha_k$ of (algebraic) multiplicities $m_1, \ldots, m_k$ summing to d. Corresponding to α_i, there is a m_i-dimensional space of (generalized) eigenvectors with generalized eigenvalue α_i. Write U_i for the space spanned by these generalized eigenvectors.

Now we have a decomposition $\mathbb{R}^d = U_1 \oplus U_2 \oplus \cdots \oplus U_k$ into subspaces with the property that if $v \in U_i \setminus \{0\}$, then

$$\tfrac{1}{n} \log \|A^{(n)} v\| \to \log |\alpha_i|.$$

Hence, at a fixed point of the dynamical system, there's a direct sum decomposition of the tangent space into subspaces, each expanding at a characteristic exponential rate.

Question. *How much of this survives if we're not taking powers of a single matrix?*

We'd like to find a decomposition of $\mathbb{R}^d$ over a point ω into subspaces expanding asymptotically at different rates.

Observations:

(1) We have different matrices over different ω's – should expect the decomposition to depend on ω.
(2) Notice asymptotically if $A_\omega^{(n)} v$ grows at a rate λ, then setting $w = A_\omega v$, $A_{\sigma(\omega)}^{(n)} w$ grows at the same rate.

We might hope that the decomposition is *equivariant*:

$$A_\omega U_i(\omega) = U_i(\sigma\omega). \tag{1}$$

The above suggests we're looking for some sort of vector bundle decomposition.

Over each ω, look for a decomposition of $\mathbb{R}^d$ as

$$\mathbb{R}^d = U_1(\omega) \oplus U_2(\omega) \oplus \cdots \oplus U_k(\omega).$$

This should be equivariant and elements of $U_i(\omega)$ should have the same exponential growth rate λ_i.

Theorem 1. *Oseledets Multiplicative Ergodic Theorem (1968)* [12]*: Invertible Form Let $(\Omega, \mathbb{P})$ be a probability space. Let $\sigma \colon (\Omega, \mathbb{P}) \to$*

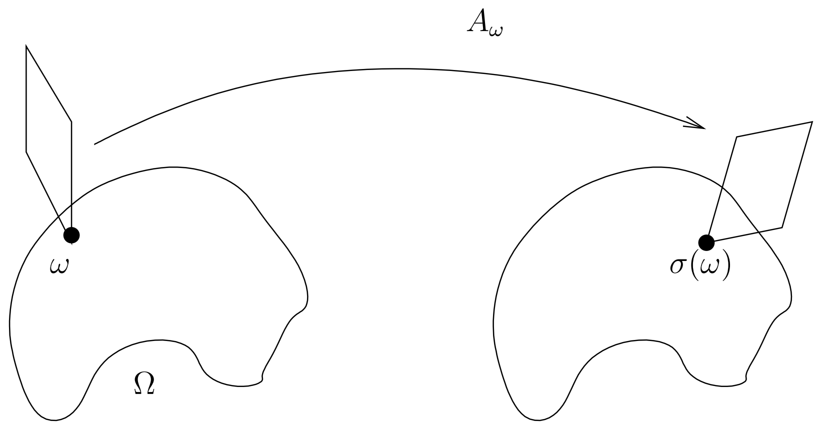


FIGURE 1. We think of A_ω as a map from the $\mathbb{R}^d$ fibre over ω to the fibre over $\sigma(\omega)$. From this viewpoint, $A_\omega^{(n)}$ is the composition of the fibre maps sending the fibre over ω to the fibre over $\sigma^n\omega$

$(\Omega, \mathbb{P})$ be invertible and ergodic. Let $A : \Omega \to GL_d(\mathbb{R})$ be measurable and satisfy ***bilateral integrability****:*

$$\int \log \|A_\omega\| \, d\mathbb{P}(\omega) < \infty$$
$$\int \log \|A_\omega^{-1}\| \, d\mathbb{P}(\omega) < \infty$$

Then there exist $\lambda_1 > \lambda_2 > \ldots > \lambda_k$ and multiplicities $m_1, \ldots, m_k$ such that $d = m_1 + \ldots + m_k$ and a measurable decomposition $\mathbb{R}^d = U_1(\omega) \oplus \ldots \oplus U_k(\omega)$ satisfying:

Equivariance:
Bilateral Growth conditions: *For $v \in U_j(\omega) \setminus \{0\}$,*

$$\tfrac{1}{n} \log \|A_\omega^{(n)} v\| \to \lambda_j \text{ as } n \to \infty$$

$$\tfrac{1}{n} \log \|A_\omega^{(n)} v\| \to \lambda_j \text{ as } n \to -\infty,$$

where $A_\omega^{(-\ell)} = A_{\sigma^{-\ell}\omega} \cdots A_{\sigma^{-1}\omega}$ so that $A_\omega^{(n+n')} = A_{\sigma^n\omega}^{(n')} A_\omega^{(n)}$ for all $n, n' \in \mathbb{Z}$.

Detailed point: When we talk about measurability of subspaces of $\mathbb{R}^d$, there is a natural metric on subspaces of $\mathbb{R}^d$ given by $d(V,V') = d_{\mathrm{Hausdorff}}(B\cap V, B\cap V')$. With this metric, the subspaces of dimension k form a connected compact metric space, the Grassmannian. The metric induces a Borel σ-algebra on the Grassmannian. As usual, measurable maps into the Grassmannian can be approximated by continuous maps into the Grassmannian.

In the non-invertible case (either the dynamical system or the matrices are not invertible – or the integrability condition fails for the inverse), the conclusion of the theorem is much weaker:

Theorem 2 (Oseledets Theorem: Non-invertible form [12])**.** *Let $(\Omega,\mathbb{P})$ be a probability space. Let $\sigma\colon(\Omega,\mathbb{P})\to(\Omega,\mathbb{P})$ be ergodic (not necessarily invertible). Let $A:\Omega\to M_d(\mathbb{R})$ be measurable and satisfy **forward integrability**:*

$$\int \log\|A_\omega\|\,d\mathbb{P}(\omega)<\infty$$

*Then there exist $\lambda_1>\lambda_2>\ldots>\lambda_k$ and multiplicities $m_1,\ldots,m_k$ such that $d=m_1+\ldots+m_k$ and a measurable **filtration** $\mathbb{R}^d=V_1(\omega)\supset V_2(\omega)\supset\cdots\supset V_k(\omega)\supset V_{k+1}(\omega)=\{0\}$ satisfying:*

Equivariance: $A_\omega V_j(\omega)\subseteq V_j(\sigma(\omega))$;
Forward Growth conditions: *For* $v\in V_j(\omega)\setminus V_{j+1}(\omega)$,

$$\tfrac{1}{n}\log\|A_\omega^{(n)}v\|\to\lambda_j \text{ as } n\to\infty.$$

This means that, for example, the λ_2 exponent is identified not with a dimension m_2 subspace, but rather a dimension $d-m_1$ subspace. Typically the m's are equal to 1. So if d is large, this is a drastic difference in the dimension.

Corollary 3. *(Uniform growth of complementary subspaces; Barreira and Silva*[2]*) Let $Z(\omega)$ be a measurable family of complementary subspaces to $V_j(\omega)$. Then for all $\epsilon>0$, for almost all ω, there exists a constant $C(\omega)$ such that for all $v\in Z(\omega)$ and all $n>0$, $\|A_\omega^{(n)}z\|\geq C(\omega)e^{(\lambda_{j-1}-\epsilon)n}\|v\|$.*

Recently, with Gary Froyland and Simon Lloyd, we proved a *semi-invertible* version of the multiplicative ergodic theorem addressing the dimension problem mentioned above.

Theorem 4 (Oseledets Theorem: Semi-invertible form)**.** *Let* $(\Omega, \mathbb{P})$ *be a probability space. Let* $\sigma\colon (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$ *be ergodic and* ***invertible****. Let* $A : \Omega \to M_d(\mathbb{R})$ *be measurable (but not necessarily invertible) and satisfy forward integrability.*

Then there's a measurable ***direct sum decomposition*** *of* $\mathbb{R}^d$ *satisfying equivariance and the forward growth condition as in the invertible case.*

2. Useful Tricks

Experts can skip to Lemma 7

2.1. Left/Right Eigenvectors.

This section motivates a phenomenon that will occur in a proof below.

Take a matrix and assume it's diagonalizable. Recall that the characteristic equation (and hence eigenvalues) of A and A^T are the same. Notice that an eigenvector of A^T is a left eigenvector of A. Now let the eigenvalues of A be $\alpha_1, \alpha_2, \ldots, \alpha_d$, the corresponding eigenvectors of A be $v_1, \ldots, v_d$; and the corresponding eigenvectors of A^T be $w_1, \ldots, w_d$. Suppose also that the multiset of eigenvalues is β_1 repeated m_1 times up to β_k repeated m_k times.

If $\alpha_i \neq \alpha_j$, we have $w_i^T A v_j = \alpha_i w_i^T v_j$ and $w_i^T A v_j = \alpha_j w_i^T v_j$, so that $w_i^T v_j = 0$.

A corollary of this is: if W is the space spanned by eigenvectors of A^T with eigenvalues $\beta_1, \ldots, \beta_j$ and V is the space spanned by eigenvectors of A with eigenvalues $\beta_{j+1}, \ldots, \beta_k$, then $W = V^\perp$.

2.2. Sub-additive ergodic theorem.

Given a collection of functions $f_n\colon \Omega \to \mathbb{R}$, they're sub-additive if they satisfy for $n, m > 0$ and for all ω,

$$f_{n+m}(\omega) \leq f_n(\omega) + f_m(\sigma^n \omega).$$

Theorem 5 (Kingman Subadditive Ergodic Theorem (1976)[8])**.** *Let* $\sigma\colon (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$ *be ergodic and measure-preserving. Let* $(f_n)_{n\in\mathbb{N}}$ *be a sub-additive sequence (with* $\int (f_1)^+ < \infty$*). Then for* $\mathbb{P}$*-almost every* ω*,*

$$\frac{f_n(\omega)}{n} \to C \text{ as } n \to \infty,$$

where $C = \inf_k \frac{1}{k} \int f_k$.

One application (of many): taking $f_n(\omega) = \log \|A_\omega^{(n)}\|$, we get almost-everywhere convergence of $(1/n)\log \|A_\omega^{(n)}\|$ to a quantity independent of ω. (This is λ_1).

Corollary 6. *Let σ, Ω, $\mathbb{P}$ and (f_n) be as in the statement of Theorem 5. Assume additionally that σ is invertible. Then $f_n(\sigma^{-n}\omega)/n \to C$ for almost every ω, where C is the constant that occurs in Theorem 5.*

Proof. Let $g_n(\omega) = f_n(\sigma^{-n}\omega)$. Then $g_{n+m}(\omega) = g_n(\omega) + g_m(\sigma^{-n}\omega)$, so that g_n is a sub-additive sequence for the ergodic transformation σ^{-1}. Hence $g_n(\omega)/n$ converges pointwise almost everywhere to $\inf_k(\int g_k)/k = \inf_k(\int f_k)/k$. $\square$

2.3. **Exterior Algebra.** The kth exterior power of a vector space V, $\Lambda^k V$ is the vector space spanned by vectors of the form $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ satisfying relations like $(17v_1) \wedge v_2 \wedge \ldots \wedge v_k = 17(v_1 \wedge v_2 \ldots \wedge v_k)$ and $v_2 \wedge v_1 \wedge v_3 \wedge \ldots \wedge v_k = -v_1 \wedge v_2 \wedge \ldots \wedge v_k$ (hence $v \wedge v \wedge v_3 \wedge \ldots \wedge v_k = 0$).

Think determinants

It's a $\binom{d}{k}$-dimensional space (a basis is $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \colon i_1 < i_2 < \ldots < i_k\}$).

If a matrix A acts on V, there's a matrix $\bigwedge^k A$ acting on $\bigwedge^k V$ just given by applying A to each coordinate of the wedge.

Using this, together with the sub-additive ergodic theorem, we can compute a sequence of expansion rates: take $f_n^k(\omega) = \log \left\| \bigwedge^k A_\omega^{(n)} \right\|$. These are sub-additive. *Define* the limit of $f_n^k(\omega)/n$ to be $\mu_1 + \ldots + \mu_k$. These are necessarily non-increasing (see Corollary 8 below).

Suppose the distinct values of the μ_i are $\lambda_1, \ldots, \lambda_k$, with λ_i occurring m_i times. Then these will turn out to be exactly the λ's appearing in the statement of the multiplicative ergodic theorem.

2.4. **Singular Value Decomposition.** Given a matrix A in $M_d(\mathbb{R})$, think of it as a linear map of $\mathbb{R}^d$. If you have a basis of $\mathbb{R}^d$ consisting of eigenvectors, then with respect to this basis, the matrix is in diagonal form. For eigenvectors, it's crucial to use the same basis in the domain and the range (they're the same space and we want to be able to iterate the map).

In our case, we think of the fibre over ω and $\sigma\omega$ as different spaces. There's no reason to use the same basis in each. Since we're choosing different bases, we would be just looking for matrices B and C such that $A = CDB^{-1}$ where D is diagonal. That's just too easy! You can take B and D to be the identity and $C = A$ for example. Now the diagonal matrix tells you nothing about A at all!

The singular value decomposition (SVD) asks to express A as a product CDB^{-1} where B and C are orthogonal. The singular values are the entries of D, the right singular vectors are the columns of B and the left singular vectors are the columns of C. It's clear (assuming uniqueness of the SVD) that A and A^T have the same singular values. Throughout, we will use the Euclidean norm on $\mathbb{R}^d$.

An inductive procedure that yields a singular value decomposition is as follows:

- Find a unit vector v_1 for which $\|Av_1\|$ is maximized;
- Assuming vectors $v_1, \ldots, v_{j-1}$ have been obtained, let R be the collection of unit vectors perpendicular to these. Choose the element of R maximizing $\|Av_j\|$.

It turns out that $Av_1, Av_2, \ldots$ are always orthogonal. To see this, one can calculate that if Av_j is not orthogonal to Av_i (suppose without loss of generality that $i < j$ and the inner product is positive), then for sufficiently small ϵ, $v_i' = (1-\epsilon^2)v_i + \epsilon v_j$ has norm less than 1, but $\|Av_i'\| > \|Av_i\|$, contradicting the choice of v_i.

Now set $w_i = Av_i/\|Av_i\|$ (if $Av_i = 0$, just pick w_i orthogonal to the previous w's). Take B to be the matrix whose columns are the v_i, C to be the matrix whose columns are the w_i and D to be the diagonal matrix with entries $\|Av_i\|$. Then one quickly checks that $A = CDB^{-1}$ as required (just apply each side to v_i).

A more streamlined procedure to find the SVD is to consider the matrix $Q = A^*A$. This is a positive semi-definite self-adjoint matrix, and hence can be diagonalized by an orthogonal matrix: $Q = BD'B^T$. We may assume without loss of generality that the diagonal entries of D' are non-increasing. Let D be the square root of D'. Create a new matrix C in the following way: For each non-zero entry of D, let the ith column of C be a normalized copy of A applied to the ith column of B. These columns are all orthogonal. For the remaining columns, simply complete C arbitrarily to an orthogonal matrix. Then $A = CDB^T$ is the required decomposition. This decomposition is essentially unique (up to ordering of singular values and choice of orthogonal basis for the blocks with a common singular value – we always assume that the singular values are arranged in decreasing order).

One can check $\|A\| = \|D\|$ and in fact $\|\bigwedge^k A\| = \|\bigwedge^k D\|$. If D has entries $z_1 \geq z_2 \geq \ldots \geq z_d$, then $\bigwedge^k D$ is diagonal with respect to the basis given earlier and has entries $z_{i_1} \cdots z_{i_k}$ where $i_1 < i_2 < \ldots < i_k$. The biggest entry is therefore $z_1 z_2 \ldots z_k$.

Writing $\chi_j(M)$ for the jth singular value of M, we obtain from the above

Lemma 7. *Let $\sigma\colon \Omega \to \Omega$ preserve the measure $\mathbb{P}$. Suppose that $\mathbb{P}$ is ergodic. Let $A\colon \Omega \to M_d(\mathbb{R})$ be measurable. Then $\| \bigwedge^k A_\omega^{(n)} \| = \chi_1(A_\omega^{(n)}) \cdots \chi_k(A_\omega^{(n)})$.*

Taking logarithms, dividing by n and taking the difference between $\log \| \bigwedge^j A_\omega^{(n)} \|$ and $\log \| \bigwedge^{j-1} A_\omega^{(n)} \|$, we obtain

Corollary 8. *Let $\mu_1, \mu_2, \ldots$ be as above. Then $\frac{1}{n} \log \chi_j(A_\omega^{(n)}) \to \mu_j$. Hence the μ_j are non-increasing in j.*

3. Multiplicative Ergodic Theorems and Duality

In this section, we give some indication of the proof of the weaker non-invertible version of the multiplicative ergodic theorem and we show how to derive the semi-invertible version from the non-invertible version. It's not hard to derive Theorem 1 from Theorem 4. The ideas for the proof we give of the non-invertible version, based on exterior algebras, singular value decomposition and the Kingman sub-additive ergodic theorem are due to Raghunathan[13]. The semi-invertible version is derived from ideas obtained in collaboration with Gary Froyland, Cecilia González Tokman and Simon Lloyd[4, 5, 16].

3.1. **The non-invertible version.**

Sketch proof of Theorem 2. The forward integrability condition allows us to apply the sub-additive ergodic theorem to $\| \bigwedge^j A_\omega^{(n)} \|$. Let the exponents (coming from the sub-additive ergodic theorem and the exterior algebra) be $\lambda_1 > \lambda_2 > \ldots > \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$.

Define spaces $V_j^n(\omega)$ to be the span of the singular vectors with the smallest $m_j + m_{j+1} + \ldots + m_k$ singular values of $A_\omega^{(n)}$.

Verify that these form an (exponentially convergent) Cauchy sequence in the Grassmannian and let $V_j(\omega)$ be the limit. Verify (using the speed of convergence which is *exactly right* for this) that if $v \in V_j(\omega)$, we have $\lim \frac{1}{n} \log \|A_\omega^{(n)} v\| \le \lambda_j$.

Let $v \in V_j(\omega) \setminus V_{j+1}(\omega)$. Write $v = u + w$ with $u \in V_{j+1}(\omega)$ and $w \in V_j(\omega) \ominus V_{j+1}(\omega)$ (here $V \ominus W$ is the orthogonal complement of W in V). For n sufficiently large, u is very close to $V_{j+1}^n(\omega)$ and w is very close to $V_j^n(\omega) \ominus V_{j+1}^n(\omega)$. We therefore obtain $\|A_\omega^{(n)} u\| \lesssim e^{(\lambda_{j+1}+\epsilon)n}$ while $\|A_\omega^{(n)} w\| \gtrsim e^{(\lambda_j - \epsilon)n}$. Hence $\|A_\omega^{(n)} v\| \ge \|A_\omega^{(n)} w\| - \|A_\omega^{(n)} u\| \gtrsim e^{(\lambda_j-\epsilon)n}\|w\|$

Hence $V_j(\omega)$ **is exactly the collection of vectors that expand at rate λ_j or less**. Equivariance follows. $\square$

3.2. Non-invertible implies semi-invertible.

Proof of Theorem 4 from Theorem 2. We are now assuming that the base dynamical system, σ is invertible. This allows us to define the *dual cocycle* over the *inverse dynamical system*, σ^{-1}. That is, one applies the map σ^{-1} each time, rather than σ. The generator of the dual cocycle is $A^*(\omega) = A(\sigma^{-1}\omega)^T$, so that $A^{*(n)}_\omega = A^*_{(\sigma^{-1})^{n-1}\omega} \dots A^*_\omega = A(\sigma^{-n}\omega)^T \dots A(\sigma^{-1}\omega)^T = (A^{(n)}_{\sigma^{-n}\omega})^T$.

By the earlier remark (that M and M^T have the same singular values) plus Corollary 6, we check that the exponents and multiplicities of the dual cocycle are the same as those of the original cocycle.

Let $V_j^*(\omega)$ be the filtration obtained for the dual cocycle. We then define $W_j(\omega) = (V_j^*(\omega))^\circ$, the annihilator of $V_j^*(\omega)$.

Lemma 9. *$W_j(\omega)$ is an equivariant complementary space to $V_j(\omega)$.*

> *$V_j(\omega)$ is the 'slow space' consisting of vectors that expand at rate λ_j or slower. $W_j(\omega)$ will be the fast space. All non-zero vectors in $W_j(\omega)$ will expand at rate λ_{j-1} or faster (that is strictly faster than λ_j).*

> *The statement of the above lemma should be reminiscent of Subsection 2.1*

A key feature is that the slow space depends on the 'future' of ω, while the fast space depends on the past of ω. That is, to decide if $v \in V_j(\omega)$, one needs to know $(A_{\sigma^n\omega})_{n\geq 0}$, whereas to decide if $v \in W_j(\omega)$, one needs to know $(A_{\sigma^n\omega})_{n<0}$. This is analogous to the definition of stable and unstable manifolds. To find the stable manifold, you look at the points that behave well in the future; to find the unstable manifold, you look at points that behave well in the past.

Right now I make the following claims:

(a) $W_j(\omega)$ is equivariant;
(b) $W_j(\omega) \cap V_j(\omega) = \{0\}$ almost surely.

Since the dual cocycle and primal cocycle have the same exponents and multiplicities, $V_j^*(\omega)$ has the same dimension (i.e. $m_j + m_{j+1} + \dots + m_k$) as $V_j(\omega)$. By linear algebra (the 'rank-nullity theorem'), the dimensions of $V_j^*(\omega)$ and $W_j(\omega)$ sum to d, so that $V_j(\omega)$ and $W_j(\omega)$ have complementary dimensions. Hence establishing (a) and (b) shows that $W_j(\omega)$ is an equivariant complement of $V_j(\omega)$ which will prove the lemma.

To prove (a), note that $A^*_\omega V^*_j(\omega) \subset V^*_j(\sigma^{-1}\omega)$ by the equivariance property of the filtration for the dual cocycle (this would be an equality if the matrices were known to be invertible as the two sides would have the same dimension). We rewrite this as

$$A^*_{\sigma\omega} V^*_j(\sigma(\omega)) \subset V^*_j(\omega). \tag{2}$$

Let $v \in W_j(\omega)$ and $\theta \in V^*_j(\sigma(\omega))$. Then

$$\langle \theta, A_\omega v\rangle = \langle A^T_\omega \theta, v\rangle = 0,$$

where the second equality follows since by (2), the left member belongs to $V^*_j(\omega)$ and the right belongs to $(V^*_j(\omega))^\circ$.

This shows that $A_\omega v$ is annihilated by any element of $V^*_j(\sigma(\omega))$ and hence establishes the inclusion

$$A_\omega(W_j(\omega)) \subset W_j(\sigma(\omega)). \tag{3}$$

The following diagram plays a key role in what comes next.

$$\begin{array}{ccccccccc}
(\mathbb{R}^d)_\omega & \xrightarrow{A_\omega} & (\mathbb{R}^d)_{\sigma(\omega)} & \xrightarrow{A_{\sigma(\omega)}} & (\mathbb{R}^d)_{\sigma^2\omega} & \xrightarrow{A_{\sigma^2\omega}} & (\mathbb{R}^d)_{\sigma^3\omega} & \xrightarrow{A_{\sigma^3\omega}} & (\mathbb{R}^d)_{\sigma^4\omega} \\
 & & & & & & & & \\
(\mathbb{R}^d)^*_\omega & \xleftarrow{A^T_\omega} & (\mathbb{R}^d)^*_{\sigma(\omega)} & \xleftarrow{A^T_{\sigma(\omega)}} & (\mathbb{R}^d)^*_{\sigma^2\omega} & \xleftarrow{A^T_{\sigma^2\omega}} & (\mathbb{R}^d)^*_{\sigma^3\omega} & \xleftarrow{A^T_{\sigma^3\omega}} & (\mathbb{R}^d)^*_{\sigma^4\omega}
\end{array}$$

The top row of the diagram consists of $\mathbb{R}^d$ fibres sitting over points on an orbit in the base, while the bottom row consists of $\mathbb{R}^d$ (considered as dual spaces to the top $\mathbb{R}^d$'s) fibres sitting over the orbit. The map A_ω maps the fibre over ω to that over $\sigma(\omega)$, whereas the transpose, A^T_ω maps the dual fibre over $\sigma(\omega)$ to that over ω.

The defining duality property ensures that if we take any point v in $(\mathbb{R}^d)_\omega$ and push it forward by $A^{(n)}_\omega$; and any dual vector θ in $(\mathbb{R}^d)^*_{\sigma^n\omega}$ and pull it back by $A^{*(n)}_{\sigma^n\omega}$, one has

$$(A^{*(n)}_{\sigma^n\omega}\theta)(v) = \theta(A^{(n)}_\omega v). \tag{4}$$

Let $Z(\omega)$ be a family of complementary subspaces to $V_j(\omega)$ as in Corollary 3 and let $C = C(\omega)$ be as in that statement. Now if $v \in Z(\omega)$ is any vector of norm 1 in $Z(\omega)$, we have $\|A^{(n)}_\omega v\| \geq Ce^{n(\lambda_{j-1}-\epsilon)}$. Now let $\theta \in V^*_j(\sigma^n\omega)$ be of norm 1. Since $\theta \in V^*_j(\sigma^n\omega)$, we have $\|A^{*(n)}_{\sigma^n\omega}\theta\| \leq C'e^{n(\lambda_j+\epsilon)}$ (where C' is independent of the choice of θ).

The next lines form the core of the argument

Hence if we let $w = A^{(n)}_\omega v$, by (4), we have $\theta(w) \lesssim e^{-n(\lambda_{j-1}-\lambda_j)}\|w\|$ and this holds for an orthonormal basis of θ's in $V^*_j(\sigma^n\omega)$. We conclude that if w is written as $w_1 + w_2$, where w_1 lies in $W_j(\sigma^n\omega) = (V_j^*(\sigma^n\omega))^\circ$ and w_2 lies in $W_j(\sigma^n\omega)^\perp$, then $\|w_2\| \lesssim e^{-n(\lambda_{j-1}-\lambda_j)}\|w\|$.

Since $A_\omega^{(n)}$ expands all vectors in $Z(\omega)$, we have $A_\omega^{(n)}Z(\omega)$ is of the same dimension, namely $m_1+\ldots+m_{j-1}$, as $W_j(\sigma^n\omega)$. Since all vectors in $A_\omega^{(n)}Z(\omega)$ lie close to $W_j(\sigma^n\omega)$ and they are of the same dimension, facts about Grassmannian imply $A_\omega^{(n)}Z(\omega)$ is exponentially close to $W_j(\sigma^n\omega)$.

On the other hand, we can show that there is a separation that is at worst sub-exponentially small (i.e. bigger than any decreasing exponential) between points of $A_\omega^{(n)}Z(\omega)$ and $V_j(\sigma^n\omega)$ (otherwise the uniform growth conditions are contradicted).

Hence we conclude that for large n, $V_j(\sigma^n(\omega)) \cap W_j(\sigma^n\omega) = \{0\}$. By the Poincaré Recurrence theorem, this must hold for all n and (b) is satisfied. □

By Corollary 3, all non-zero vectors in $W_j(\omega)$ have expansion rate λ_{j-1} or faster.

We now use the spaces $W_j(\omega)$ and $V_j(\omega)$ to give a swift conclusion to the argument. In fact, we define $U_j(\omega) = V_j(\omega)\cap W_{j+1}(\omega)$ and claim that

(1) $U_j(\omega)$ is equivariant;
(2) $U_1(\omega) \oplus U_2(\omega) \oplus \cdots \oplus U_k(\omega) = \mathbb{R}^d$.

The first of these is immediate since it is the intersection of two equivariant sets. For the second, we show that

$$V_j(\omega) = V_{j+1}(\omega) \oplus U_j(\omega),$$

from which the fact that the $U_j(\omega)$ form a decomposition follows inductively. To see this, first notice that $V_{j+1}(\omega) \cap U_j(\omega) \subset V_{j+1}(\omega) \cap W_{j+1}(\omega) = \{0\}$ from above. Secondly, if $v \in V_j(\omega)$, then since $W_{j+1}(\omega)$ is a complementary space of $V_{j+1}(\omega)$ in $\mathbb{R}^d$, we have $v = w + z$ where $w \in W_{j+1}(\omega)$ and $z \in V_{j+1}(\omega)$. Now since v and z both lie in $V_j(\omega)$, it follows that $w \in V_j(\omega)$ and hence $w \in U_j(\omega)$. This establishes $V_j(\omega) = V_{j+1}(\omega) \oplus W(\omega)$ as required.

Note that in the case where the matrices are invertible, there is a similar, but alternative route used by Oseledets and implicitly by Raghunathan. In place of the dual cocycle, one can study the *inverse* cocycle over σ^{-1} generated by $B_\omega = A^{-1}_{\sigma^{-1}\omega}$. In this case, no duality is needed. The exponents for the inverse cocycle are the negatives of those for the primal cocycle. To find $U_j(\omega)$, one intersects $V_j(\omega)$ with $V^{-1}_{d-j-1}(\omega)$, that is the set of vectors that are no faster than λ_j in the future, together with the set of vectors that are no faster than $-\lambda_j$ in the past.

In the case where the matrices are non-invertible, the inverse cocycle can obviously not be used, so the advantage of using the dual cocycle is that one obtains a decomposition of the space rather than a filtration, even if the matrices are non-invertible.

3.3. Multiplicative Ergodic Theorem Extensions.

The multiplicative ergodic theorem has been extended in many ways by many people. One direction of this is the generalization of matrices to linear operators. Results in this direction are due to Ruelle[14] (compact operators on Hilbert space), Mañé[10] (compact operators on Banach spaces), Thieullen[15] (norm-continuous families of quasi-compact operators on Banach spaces) and Lian and Lu[9] (families of operators on separable Banach spaces continuous in the "strong operator topology" [a topology weaker than the norm topology!]).

Mañé's paper, on which that of Thieullen and the book of Lian and Lu were based was in the context of injective operators (without the assumption of invertibility). For them, the inverse cocycle was therefore not defined.

Thieullen introduced a black box which deduced a non-invertible form from the invertible form. The proof above that non-invertible yields semi-invertible generalizes to the settings of Thieullen and Lian and Lu.

Question. *Can the Raghunathan method described above be used to give streamlined versions of* non-invertible *versions of operator multiplicative ergodic theorems?*

If so, the above techniques allow an easy extension to semi-invertible; and possibly to injective versions (the difference being that the latter control backwards rates of growth also).

4. Perron-Frobenius Operators and Ulam's Method

4.1. Perron-Frobenius Operators.

Given a map T from a space (X, m) to itself (T is not assumed to preserve the measure m), it's said to be *non-singular* if $m(T^{-1}B) = 0$ whenever $m(B) = 0$. The above is an obvious necessary condition to be able to find an invariant measure equivalent to m.

Given a non-singular map (a simple example to which we shall return often is the case of a piecewise expanding map of the interval), one often wants to look for an equivalent absolutely continuous measure. A key tool for this is the *Perron-Frobenius operator*.

If $\rho(x)$ is a non-negative L^1 function of integral 1 (a density), then the Perron-Frobenius operator, $\mathcal{L}$ can be thought of as follows:

If one picks a point from the space with distribution density $\rho(x)$ and apply T to it, the distribution density of the image point is given by $\mathcal{L}[\rho](x)$

A consequence of this is: if one computes the expected value of $f \circ T$ with respect to the density ρ, that is $\int \rho(x) f \circ T(x)\, dm(x)$, this should be the same as the expected value of f with respect to the density $\mathcal{L}\rho$, $\int f(y)\mathcal{L}\rho(y)\, dm(y)$.

In fact, Perron-Frobenius operators are characterized by the equality

$$\int f(T(x))g(x)\, dm(x) = \int f(x)\mathcal{L}g(x)\, dm(x),$$

where f is an arbitrary L^∞ function and g an L^1 function.

Notice that if $\mathcal{L}g = g$, then $g\, dm$ is an invariant measure, absolutely continuous with respect to m. There is a one–one correspondence between fixed points of the Perron-Frobenius operator and absolutely continuous invariant measures.

In nice situations (like transitive piecewise smooth expanding maps), there is a unique absolutely continuous invariant measure with density g_0 and one has $\mathcal{L}^n g \to (\int g\, dm)g_0$ in the uniform norm.

Given this, one can conclude a decay of correlations result:

$$\begin{aligned}\int f \circ T^n g\, dm &= \int \mathcal{L}^n g(x) f(x)\, dm(x)\\ &\to \int \left(\int g\, dm\right) g_0(x) f(x)\, dm(x)\\ &= \int g\, dm \int f \cdot g_0\, dm.\end{aligned}$$

In particular, if m is an invariant measure (so that $g_0 = 1$), this gives

$$\int f \circ T^n g\, dm \to \int f\, dm \int g\, dm,$$

an asymptotic independence (or mixing) result.

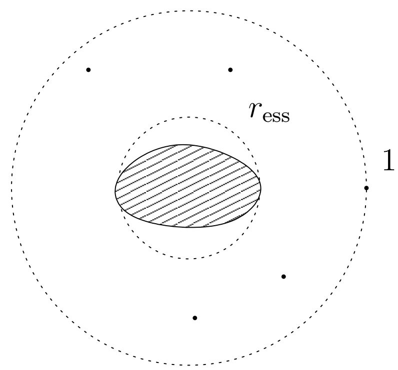
One might ask for the *rate* of convergence. Unfortunately, this can be arbitrarily slow for general $g \in L^1$.

You can build a counterexample by noting that $\mathcal{L}(h \circ T) = h$ when the invariant density, g_0 is 1, and taking g something like $\sum(1/2^n)h \circ T^{n!!!}$ for a function like $h = \sin x$ – this example is something like the standard continuous nowhere differentiable function on steroids.

Arguments similar to this show that the spectrum of $\mathcal{L}$ is the entire unit disk.

One *can* sometimes obtain a rate of convergence if the Banach space is restricted from L^1 to a finer Banach space $B \subset L^1$ (with a stronger norm also). Of course one needs $\mathcal{L}(B) \subset B$.

A well known candidate in the interval case is the space BV, of functions of bounded variation. In that case, it's known that the spectrum of $\mathcal{L}$ looks something like



The spectral point at 1 corresponds to the fixed point of $\mathcal{L}$. The other points correspond to other eigenfunctions of $\mathcal{L}$ and the blob in the middle is the 'essential spectrum'. The spectral radius is the maximum absolute value of a point in the spectrum and the essential spectral radius is the maximum absolute value of a point in the essential spectrum.

An operator for which the essential spectral radius is strictly smaller than the spectral radius is called *quasi-compact* (as the operator shares a number of the properties of compact operators on Banach spaces).

In this case (provided g belongs to BV), the decay of correlations is exponential, governed by the second eigenvalue of the $\mathcal{L}$ acting on BV.

The Lasota-Yorke inequality for a piecewise expanding C^2 map of the interval (for the Banach space pair (BV,L^1)) is

$$\|\mathcal{L}g\|_{\text{BV}} \leq \frac{2}{\min|T'|}\|g\|_{\text{BV}} + C\|g\|_1.$$

Given an inequality of the form $\|\mathcal{L}f\|_{\text{BV}} \leq \alpha\|f\|_{\text{BV}} + \beta\|f\|_1$, α can be shown to be an upper bound for the essential spectral radius (this is a result of Hennion[6] using a characterization of the essential spectral radius due to Nussbaum[11]). In this case, better bounds can be obtained by considering T^n instead of T.

4.2. **Ulam's Method.** Ulam proposed in his book [17] in the 1960's a very naive method of computing invariant measures of dynamical systems. Suppose the dynamical system is T, acting on a space X. Divide X up into small pieces $I_1, I_2, \dots, I_N$. Compute the matrix P with entries $P_{ij} = m(I_i \cap T^{-1} I_j)/m(I_i)$. This computes the conditional probability that $T(x)$ lies in I_j given that x lies in I_i with distribution $m|_{I_i}$. Now pretend that the dynamics of T is actually the same as that given by the Markov chain with transition matrix P. Compute the left eigenvector π with eigenvalue 1 for P and deduce an approximation to the invariant density for T.

Remarkably this works quite well, at least for expanding dynamical systems!

The approximation to the fixed point by Ulam's method may be re-expressed as the fixed point of the finite rank operator $\tilde{\mathcal{L}} = \Pi \circ \mathcal{L}$, where Π is the conditional expectation operator $\Pi(f) = \mathbb{E}_m(f|\mathcal{P})$, where $\mathcal{P}$ is the partition of the space.

If $\mathcal{L}f = f$ and $\tilde{\mathcal{L}}\tilde{f} = \tilde{f}$, then we're interested in $\|f - \tilde{f}\|_1$. We estimate as follows:

$$\begin{aligned}
\|\tilde{f} - f\|_1 &= \lim_{n\to\infty} \|\tilde{\mathcal{L}}^n f - f\|_1 \\
&\le \sum_{n=0}^{\infty} \|\tilde{\mathcal{L}}^{n+1} f - \tilde{\mathcal{L}}^n f\|_1 \\
&= \sum_{n=0}^{\infty} \|\tilde{\mathcal{L}}^n (\Pi\mathcal{L}f - f)\|_1 \\
&= \sum_{n=0}^{\infty} \|\tilde{\mathcal{L}}^n (\Pi - I) f\|_1.
\end{aligned}$$

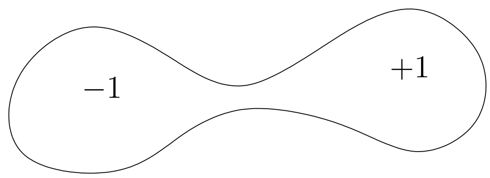
$(\Pi - I)f$ is a function of L^1 norm approximately $1/n$ (but integral 0). Let us study the interval case. On each sub-interval, it is negative in one part and positive in another. It takes roughly $\log n$ steps until these are spread over the whole interval. After this time, there is exponential decay. Hence one obtains a heuristic estimate $\|\tilde{f} - f\|_1 \approx \log n/n$. This can be established rigorously and is known to be sharp.

A finer analysis was carried out by Keller and Liverani[7], who showed that not only does the Ulam method approximate the invariant density, but also subsequent eigenvectors of $\tilde{\mathcal{L}}$ approximate those of $\mathcal{L}$ for eigenvalues outside the essential spectral radius.

4.3. **Dellnitz-Froyland Ansatz.** Dellnitz, Froyland and Sertl [3] studied the peripheral spectrum of the Perron-Frobenius operator (the part of the spectrum lying outside the essential spectrum, and consisting of at most countably many isolated points, each an eigenvalue with finite-dimensional eigenspace).

They noticed that when a dynamical system consists of a number of almost-invariant pieces, these pieces are often detected by eigenfunctions.

A cartoon of this is as follows:



Given an eigenfunction with eigenvalue close to 1, sublevel sets and superlevel sets are used to locate almost invariant regions.

This has been used to locate poorly mixing regions of the ocean (see Figure 2).

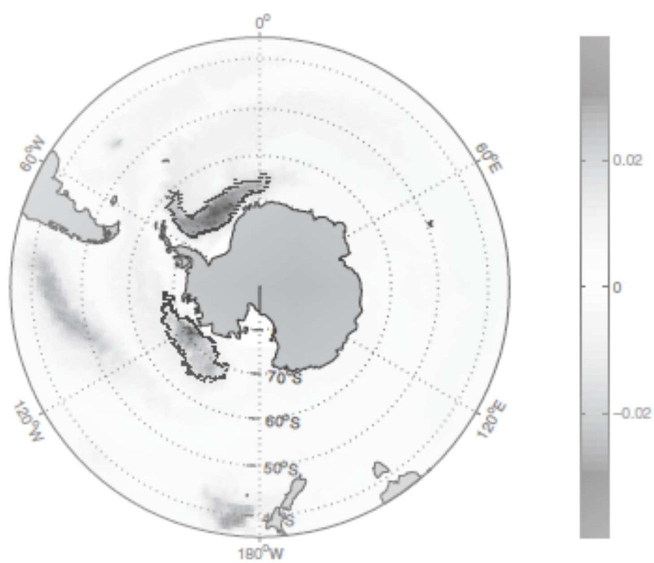


FIGURE 2. Gyres in the Southern Ocean. The central region is Antarctica. The smaller dark parts around Antarctica are gyres (slowly mixing regions of the ocean), as calculated using Ulam's method and the Dellnitz-Froyland Ansatz.

5. Rates of mixing in Forced Dynamical Systems

The underlying goal of our program of research has been to find an extension of the Ulam and Dellnitz-Froyland machinery, which work for a single dynamical system, to the context of forced dynamical systems.

We have an ergodic measure-preserving base dynamical system $\sigma\colon \Omega \to \Omega$. This could be a Bernoulli shift (if the maps are to be chosen randomly), an irrational torus rotation (if one wants to study a quasi-periodic case) etc.

For each ω, there's a map T_ω of a space X to itself. We will study these through their Perron-Frobenius operators, $\mathcal{L}_\omega$.

As discussed previously, it no longer makes sense to look for eigenvalues and eigenfunctions. In their place, we'll look for exponents and the corresponding equivariant families of subspaces. We are therefore talking about a operator-valued multiplicative ergodic theorems. We make use of the theorems of Thieullen; and Lian-Lu.

In slightly more detail, (versions of) these theorems are:

Theorem 10. *(Thieullen – injective* [15]*) Let $\sigma\colon \Omega \to \Omega$ be a homeomorphism of a compact metric space with an ergodic invariant measure $\mathbb{P}$. Let X be a Banach space and let $(\mathcal{L}_\omega)_{\omega\in\Omega}$ be an (almost) continuously parameterized (in the operator norm) family of injective operators satisfying a quasi-compactness condition.*

Then there exist $\lambda_1 > \lambda_2 > \ldots > \lambda_\infty = \kappa$, multiplicities $m_1, m_2, \ldots$ and equivariant families of m_i-dimensional spaces $U_i(\omega)$ and $R(\omega)$ such that $X = U_1(\omega) \oplus U_2(\omega) \oplus \ldots \oplus R(\omega)$. Elements of $U_i(\omega)$ have backward and forward growth rate λ_i. Elements of $R(\omega)$ have forward growth rate at most κ.

A major development here over the previous Mañé result was the introduction of a suitable version of quasi-compactness in this context.

Theorem 11. *(Thieullen – non invertible* [15]*) No invertibility assumption on the dynamics; No injectivity condition. Conclusion: replace decomposition with a filtration. No backward growth rate*

Perron-Frobenius operators of non-invertible systems are essentially never injective, so we could not directly apply the invertible version. But the conclusion of the non-invertible version gives a filtration. The second equivariant subspace is typically a 1-codimensional space :(

We found a semi-invertible version. Another difficulty for us is that the condition $\omega \mapsto \mathcal{L}_\omega$ is (almost) continuous in the operator norm is extremely strong. In fact, such a map has at most countable range. This gave a result for a forced countable family of interval maps. (Froyland + Lloyd + Quas)

Theorem 12. *(Lian and Lu – invertible* [9]*) Roughly the same setup; now X is a* separable *Banach space. The major advantage for us is the continuity assumption is greatly weakened. Now we require for any* fixed *g, $\omega \mapsto \mathcal{L}_\omega g$ is continuous. Same conclusion.*

Doan, in his PhD thesis used Thieullen's invertible→non-invertible machine to create a non-invertible version of Lian and Lu's work. The machine described earlier creates a semi-invertible version.

Viviane Baladi and Sébastien Gouëzel [1] created a family of fancy Sobolev Banach space on which Perron-Frobenius operators of certain piecewise hyperbolic maps are quasi-compact. They happen to be separable and compatible with the required continuity condition!

As a corollary, we can now prove the existence of Oseledets spaces for a continuous family of expanding maps in finite dimension subject to a bounded complexity condition.

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