ON REPRESENTATIONS OF MARKOV CHAINS BY RANDOM SMOOTH MAPS

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ABSTRACT

In this paper, we consider the problem of representing a Markov chain on a smooth manifold by a measurable collection of smooth maps, and establish sufficient conditions for such a representation to exist.

1. *Introduction*

We first introduce some notation. By smooth, we always mean C^∞. A Markov chain on (M, Ω), a measurable space, will be given by a map $P: M \times \Omega \to I$, with $P(x, A)$ a measurable function of x for fixed $A \in \Omega$ and $P(x, \cdot)$ a probability measure on (M, Ω) for each x. $P(x, A)$ specifies the probability of moving from the point x into the set A. A *representation* of P is a collection $\mathscr{F}$ of maps from M to itself and a probability measure m on them such that

$$P(x, A) = m(\{f \in \mathscr{F} : f(x) \in A\}), \quad x \in M, \ A \in \Omega. \tag{1}$$

Then, as described in [5, §1.1], the Markov chain can be reconstructed by making each transition the result of picking a map from $\mathscr{F}$ with probability distribution m. In [5] Kifer goes on to show the following.

THEOREM 1. *If M is a Borel subset of a complete metric space, then any Markov chain on M can be represented by a collection of measurable maps.*

With the notation that $P_x(A) \equiv P(x, A)$, he then reproduces the result of [4] showing the following.

THEOREM 2. *Let M be a connected and locally connected compact metric space. Let P be a Markov chain on M with the properties that P_x depends continuously on x in the weak*-topology on the set of measures on M, and P_x has full support for each x. Then P may be represented by a collection of continuous maps on M.*

We consider the case when M is a smooth manifold, and P a Markov chain on M. Under certain further conditions, P may be represented by a measurable

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collection of smooth maps on M. Specifically, we take M to be a smooth, compact, orientable Riemannian manifold, with metric g. This induces a natural volume element ω, with associated Riemannian measure V, say. Let Ω be the σ-algebra of Borel sets on M, and let P be the map describing the Markov chain. The conditions on P are as follows.

$$\begin{aligned} &\text{(i) } P_x \text{ is absolutely continuous with respect to } V,\ x \in M. \\ &\text{(ii) } h(x,y) = \frac{dP(x,y)}{dV(y)} \text{ is smooth in } x \text{ and } y,\ x,y \in M. \\ &\text{(iii) } h(x,y) > 0,\ x,y \in M. \end{aligned} \tag{2}$$

The theorem is then as follows.

THEOREM 3. *Suppose that M is a smooth, compact, orientable Riemannian manifold. If P is a Markov chain on M satisfying the conditions in* (2), *then we can represent P by a collection of smooth maps on M.*

2. *Physical sketch of the proof*

We first present an outline of the proof, showing the fluid dynamical motivation. This is not essential for what follows.

The Markov chain is to be represented by a collection of smooth maps. We regard the function $h(x,y)$ as giving the density of maps taking x into a neighbourhood of y (that is, the measure of the maps taking x into a neighbourhood U of small diameter about y is approximately $h(x,y)\cdot \mathrm{Vol}(U)$). The problem is then to find a collection of maps, and a measure on them, such that the density of the images of x under the maps is $h(x,y)$. We are thus seeing the images of x for the varying maps as part of a continuum, and are seeing how the points of the continuum move as we vary x along smooth paths. Since $h(x,y) > 0$ for each $x, y \in M$, we expect to find at least one map taking any given $x \in M$ to any given $y \in M$. Further, when x moves along any smooth curve (to x' say), we expect the images of the maps to move along curves of the flow, so that if two maps agree at x, then they agree at x', and hence everywhere. With this in mind, we impose that there should be exactly one map taking each x to each y. Fixing $x_0 \in M$, each map on the manifold may thus be labelled by the image of x_0 under that map. The maps are then smooth maps f_y with the property that $f_y(x_0) = y$. We then define the map $\alpha_x : y \mapsto f_y(x)$. By the fluid analogy again, we expect the map α_x to be a smooth diffeomorphism, since $\alpha_x(y)$ is the point to which $y = \alpha_{x_0}(y)$ flows as x moves along a path from x_0 to x.

Take Π to be the space of smooth positive density distributions on the manifold (that is, smooth functions with $\int f\,dV = 1, f > 0$). Then the diffeomorphisms α on the manifold act naturally on Π as

$$\alpha^* : \Pi \longrightarrow \Pi, \quad (\alpha^*(\rho))(\alpha(x)) = \rho(x)/\text{expansion coefficient},$$

where the expansion coefficient is the absolute value of the Jacobian of the map α evaluated at the point x. This is just an expression of conservation of mass. For each $x, y \in M$, write $\rho_x(y) = h(x,y)$. Then $\rho_x \in \Pi$, and let ρ_0 be the distinguished density ρ_{x_0}.

We then define the corresponding measures μ_x by $\mu_x(A) = P(x,A)$ (the correspondence being $d\mu_x/dV = \rho_x$), and set $\mu(A) = P(x_0, A)$. We are then forced to define $m(\{f_y : y \in A\}) = \mu(A)$ by considering equation (1) in the case $x = x_0$. Further, by considering equation (1), we see that

$$P(x, A) = m(\{f_y : f_y(x) \in A\}) = \mu(\{y : f_y(x) \in A\}) = \mu(\{y : \alpha_x(y) \in A\}) = \mu(\alpha_x^{-1}(A))$$

or, since α_x is a homeomorphism, $P(x, \alpha_x(A)) = \mu(A)$. This is equivalent to saying that $\alpha_x^*(\rho_0) = \rho_x$. The problem is then reduced to finding a smoothly parametrised collection of diffeomorphisms α_x such that $\alpha_x^*(\rho_0) = \rho_x$.

It is clearly sufficient to find a collection of diffeomorphisms α_ρ such that $\alpha_\rho^*(\rho_0) = \rho$, with enough smoothness that α_{ρ_x} is smoothly parametrised by x. Now, given a $\rho \in \Pi$, define a path in Π by $\rho(t) = \rho_0 + t\eta$, where η is given by $\rho - \rho_0$. We then seek a collection $\alpha_{\rho(t)}$ of diffeomorphisms associated to densities $\rho(t)$ (that is, such that $\alpha_{\rho(t)}^*(\rho_0) = \rho(t)$). Moving along this path, there is a constant rate of change of density at each point of the manifold, such as could arise from a constant flux (by comparison with the fluid dynamics equation $\nabla \cdot \mathbf{\Phi} + \dot{\rho} = 0$, where $\mathbf{\Phi} = \rho\mathbf{v}$ is the flux). We therefore seek a flux vector field whose divergence is $-\eta$, and which depends with sufficient smoothness on η. This then gives an expression for the velocity of each point in the continuum which gives rise to the required flux (at a specific time on the path being given by $\mathbf{\Phi}/\rho(t)$), and so we let $\alpha_\rho(x)$ be the position of the point x after unit time flow along the parametrised velocity field. We shall then find that $\alpha_\rho^*(\rho_0) = \rho$, as required, and it will remain to check that we have the required smoothness. This is then shown by the theory of elliptic partial differential equations on manifolds, completing the proof.

3. *Differential equations background*

For the proof of Theorem 3, we shall need to use a lemma which relies on the following theorems from the theory of Green's functions for the Laplacian on compact manifolds.

The Laplacian is defined by $\Delta f = \nabla_i \nabla^i f$, in local coordinates, where ∇ is the covariant derivative operator on M (with the Riemannian connection).

THEOREM 4. *Let M be a smooth, compact, orientable Riemannian manifold. If f is a smooth function on M, with $\int f\,dV = 0$, then there exists u with $\Delta u = f$. Further, u is smooth and unique up to a constant.*

Proof. See [**1**, §4.1.2].

THEOREM 5. *Let M be a smooth, compact Riemannian manifold. There exists $G: M \times M \to \mathbb{R}$ such that for ϕ a smooth function on M, we have*

$$\phi(x) = V(M)^{-1} \int_M \phi(y)\,dV(y) + \int_M G(x, y)\,\Delta\phi(y)\,dV(y), \tag{3}$$

$$G(x, y) \geqslant 0 \quad \textit{for all } x, y \in M, \tag{4}$$

$$\int_M G(x, y)\,dV(y) = C, \quad \textit{a constant} < \infty, \tag{5}$$

$$G(x, y) = G(y, x). \tag{6}$$

Proof. See [**1**, §4.2.3].

Define

$$\Pi = \left\{ f\colon M \longrightarrow \mathbb{R}^+ \backslash \{0\} \text{ smooth with } \int f\,dV = 1 \right\},$$

$$\mathscr{Z} = \left\{ f\colon M \longrightarrow \mathbb{R} \text{ smooth with } \int f\,dV = 0 \right\},$$

$$\mathscr{X} = \{\text{smooth vector fields on } M\}.$$

LEMMA. *Given a smooth, compact, orientable Riemannian manifold M, and a collection η_β ($\beta \in M$) of smoothly parametrised functions in $\mathscr{Z}$ (that is, $(\beta, x) \mapsto \eta_\beta(x)$ is a smooth map $M \times M \to \mathbb{R}$), there is a map $\Phi\colon \mathscr{Z} \to \mathscr{X}$ satisfying*
(i) $\operatorname{div}(\Phi(\eta_\beta)) = -\eta_\beta$,
(ii) *the map $(\beta, x) \mapsto \Phi(\eta_\beta)(x)$ is smooth.*

Proof. Suppose that U is an open set in $\mathbb{R}^k$ and θ_α ($\alpha \in U$) is a smoothly parametrised collection of functions in $\mathscr{Z}$, then let $H_\alpha(x)$ be the solution of the equation $\Delta H_\alpha = \theta_\alpha$ such that $\int H_\alpha\,dV = 0$. This exists, by Theorem 4, and is smooth in x. Then by (3), we see

$$H_\alpha(x) = \int_M G(x, y)\,\theta_\alpha(y)\,dV(y).$$

We now have to show that the function $H(\alpha, x) \equiv H_\alpha(x)$ is smooth. The ith parametric partial derivative of H is given by

$$\begin{aligned}\frac{\partial}{\partial \alpha^i} H(\alpha, x) &= \lim_{t \to 0} \int_M \frac{1}{t} G(x, y)(\theta_{\alpha + te_i}(y) - \theta_\alpha(y))\,dV(y)\\ &= \lim_{t \to 0} \int_M G(x, y)\left(\frac{\partial}{\partial \alpha^i}\theta_\alpha(y) + \zeta(t, x)\right) dV(y),\end{aligned}$$

where e_i is the ith coordinate vector field in U and $\zeta(t, x)$, the remainder term, is smooth in x, and $\zeta(t, x) \to 0$ as $t \to 0$, for all $x \in M$. It follows that $\zeta(t, x) \to 0$ uniformly on M (by compactness) as $t \to 0$, and hence by (4) and (5), it follows that

$$\frac{\partial}{\partial \alpha^i} H(\alpha, x) = \int_M G(x, y) \frac{\partial}{\partial \alpha^i} \theta_\alpha(y)\,dV(y).$$

But the ith partial derivative of θ_α remains a smoothly parametrised collection of functions, and the ith partial derivative of H is clearly continuously dependent on α, and is a smooth function of x by the argument above, so replacing θ_α by its partial derivative in the above procedure shows inductively that $H(\alpha, x)$ depends smoothly on α and x.

Finally, take a chart (U, ψ) of M, and use the above, with $\theta_\alpha = -\eta_{\psi^{-1}(\alpha)}$, to obtain a smooth $H(\alpha, x)$ such that $\Delta H_\alpha(x) = -\eta_{\psi^{-1}(\alpha)}$. Then set $F_\beta(x) = H_{\psi(\beta)}(x)$ (chart-independent), and patch these together using the independence, to obtain a smooth function $F\colon M \times M \to M$ such that $\Delta F_\beta(x) = -\eta_\beta(x)$, and take (in local coordinates)

$$\Phi(\eta_\beta)^i(x) = \nabla^i F_\beta(x).$$

This gives the map Φ, as required.

4. *Main proof*

Proof of Theorem 3. First note that Π is a convex set, and that there is a canonical map $M \to \Pi$ given by $x \mapsto \rho_x$, where ρ_x is defined by

$$\rho_x(y) = h(x, y).$$

Let Φ be as defined in the Lemma, then define $\gamma_x(y, t)$ by

$$\gamma_x(y, 0) = y \quad \text{for all } x, y \in M, \tag{7}$$

$$\frac{d\gamma_x(y,t)}{dt} = \frac{\Phi(\rho_x - \rho_0)}{(1-t)\rho_0 + t\rho_x}(\gamma_x(y,t)).$$

Write $\gamma_{x,t}(y) \equiv \gamma_x(y,t)$ and, using the Lemma, we see that the vector field above depends smoothly on the parameters x and t, so that we can use the parametrised flow theorem (see [**2**, §21.4]) to show that the $\gamma_{x,t}$ form a smoothly parametrised collection of diffeomorphisms. In particular, define smoothly parametrised diffeomorphisms

$$\theta_x(y) = \gamma_{x,1}(y). \tag{8}$$

We shall then show that

$$\int_{\theta_x(A)} \rho_x(y)\, dV(y) = \int_A \rho_0(y)\, dV(y) \tag{9}$$

for Borel sets A and $x \in M$. Assuming this for now, we complete the claim by setting

$$f_y(x) = \theta_x(y),$$
$$\mathscr{F} = \{f_y : y \in M\},$$
$$m(\{f_y : y \in A\}) = P(x_0, A) \quad \text{for } A \in \Omega.$$

Then we check

$$P(x, A) = \int_A \rho_x(y)\, dV(y) = \int_{\theta_x^{-1}(A)} \rho_0(y)\, dV(y) = P(x_0, \theta_x^{-1}(A))$$
$$= P(x_0, \{y : \theta_x(y) \in A\}) = P(x_0, \{y : f_y(x) \in A\}) = m(\{f_y : f_y(x) \in A\})$$

using, for the second equality, (9) and the fact that θ_x is a homeomorphism. This statement is then the required condition (1), thus completing the proof subject to the proof of (9).

To prove (9), note that it is sufficient to prove it for A an open set in M with piecewise smooth boundary. So fix U open in M, take $x \in M$ and set $\rho(t) = (1-t)\rho_0 + t\rho_x$. We show that the following equation holds:

$$\frac{d}{dt}\left(\int_{\gamma_{x,t}(U)} \rho(t)(y)\, dV(y)\right) = 0. \tag{10}$$

Then (9) will follow from (7), (8) and (10).

Now, set $\eta = \rho_x - \rho_0$ and $X = \Phi(\eta)$ as defined in the Lemma. Using the transport theorem with mass density (see [3, §8.2.1]), the left-hand side of (10) is equal to

$$\int_{\gamma_{x,t}(U)} \frac{d\rho(t)}{dt}\omega + \mathscr{L}_{X/\rho(t)}(\rho(t)\,\omega),$$

where $\mathscr{L}_Z$ denotes the Lie derivative along the vector field Z. The integrand is then equal to

$$\eta\omega + \omega\mathscr{L}_{X/\rho(t)}(\rho(t)) + \rho(t)\,\mathscr{L}_{X/\rho(t)}(\omega) = \eta\omega + \frac{\omega}{\rho(t)}\cdot\left(X^i\frac{\partial}{\partial x^i}\rho(t)\right) + \rho(t)\,\mathrm{div}\left(\frac{X}{\rho(t)}\right)\omega$$

$$= \eta\omega + \frac{\omega}{\rho(t)}\cdot\left(X^i\frac{\partial}{\partial x^i}\rho(t)\right) + \mathrm{div}\,(X)\,\omega + X^i\rho(t)\,\omega\nabla_i\left(\frac{1}{\rho(t)}\right) = 0.$$

This completes the proof.

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